Modal Analysis for Controlling Elastic Waves in Platonic Metamaterials

B. Vial[1], **M. Martí Sabaté**[1], **S. Guenneau**[2] **and R. V. Craster**[3]

[1]Department of Mathematics, Imperial College London, London SW7 2AZ, UK
[2]The Blackett Laboratory, Department of Physics, Imperial College London, London SW7 2AZ, UK
[3]Department of Mathematics, UMI 2004 Abraham de Moivre-CNRS, Department of Mechanical Engineering, Imperial College London, London SW7 2AZ, UK
b.vial@imperial.ac.uk

Abstract – **We present our recent work on structured arrays and clusters of scatterers in elasticity to design elastic platonic metamaterials that utilise resonant phenomena. Numerical simulations based on a Green's function approach allows us to solve for the multiple scattering problem for resonators atop a thin elastic plate and to find eigenmodes of those open structures, the so-called quasi-normal modes. We derive a modal expansion of the displacement field that gives a valuable insight on the resonant interaction of external forces with the system's eigenmodes and that provides a reduced-order model capable of efficiently computing physical quantities of interest such as the local density of states. Potential applications of the elastic devices include elastic delay lines and passive energy harvesters.**

I. Introduction

Resonant interaction is a central concept in any system supporting wave propagation. For open structures where the resonating elements couple to an open, infinite background medium, the system becomes non-Hermitian and eigenfrequencies ω_n are generally complex, even in the absence of dissipation. The associated eigenmodes have been referred to using various terminology in the literature: quasi-normal modes (QNMs), resonant states, leaky modes, scattering resonances, or quasi bound states in the continuum for the particular case of resonances with high quality factor $Q_n = -\mathrm{Re}\,\omega_n/2\mathrm{Im}\,\omega_n$. An unintuitive feature of those QNMs is an exponential growth in the far field because of the imaginary part of the resonance frequency, a consequence of the non-hermiticity of the spectral problem. QNMs have been extensively studied in photonics for gaining physical insights on light-matter interaction and as a reduced-order model for an efficient approximation of the solution to a forced problem by expanding the field on a few modes. For elastic waves, fewer studies have been reported which include thin plates with clusters or rigid pins [1], ring arrays of masses [2] or mass-spring resonators [3] and full-elasticity in two-dimensions [4].

In this contribution we highlight how eigen-analysis of open systems in elastic structures can provide a valuable insight on the resonant properties of those systems. Based on the Green's function approach, we solve for the resulting nonlinear eigenvalue problem with an iterative method to find the complex resonant frequencies for finite clusters of scatterers (pins, masses or mass-spring resonators) placed on an elastic plate described by Kirchhoff-Love theory. We develop a quasi-normal mode expansion which reveals the contribution of each mode to the scattering properties of the system and turns out to be a computationally efficient tool to calculate the response of the structure to an arbitrary excitation when the forcing parameters (frequency, plane wave incident angle, point source position) vary.

II. Theory

Let us consider an elastic plate of thickness h, mass density ρ, Young's modulus E and Poisson ratio ν, loaded with a finite distribution of N mass-spring resonators with force constant $k_{R\alpha}$ and mass $m_{R\alpha}$, located at positions $\boldsymbol{R}_\alpha$. According to Kirchhoff-Love theory, the governing equation of motion for the plate's displacement W in time-harmonic regime $\exp(-i\omega t)$ is given by [5]:

$$\mathcal{P}(\omega)W(\boldsymbol{r}) = \left(\nabla^4 - k^4 - \mathcal{R}(\omega)\right) W(\boldsymbol{r}) = 0, \tag{1}$$

where we introduced the biharmonic operator ∇^4, the wave number k satisfying $k^4 = \omega^2\rho h/D$, the plate bending stiffness $D = Eh^3/12\left(1-\nu^2\right)$ and the operator $\mathcal{R}(\omega)$ defined as $\mathcal{R}(\omega)W(\boldsymbol{r}) = \sum_\alpha t_\alpha(\omega)W(\boldsymbol{R}_\alpha)\delta(r-\boldsymbol{R}_\alpha)$. The quantities $t_\alpha(\omega) = \frac{m_{R\alpha}}{D}\frac{\omega_{R\alpha}^2\omega^2}{\omega_{R\alpha}^2-\omega^2}$ are the resonators' strength or impedance, with $\omega_{R\alpha} = \sqrt{k_{R\alpha}/m_{R\alpha}}$ the resonant frequencies.
Solving equation (1) can be done by first finding the Green's function of the plate without resonators satisfying $\left(\nabla^4 - k^4\right)G(\boldsymbol{r}) = \delta(\boldsymbol{r})$, whose solution is known explicitly as $G(\boldsymbol{r}) = \frac{i}{8k^2}\left[H_0(kr) - H_0(ikr)\right]$, where H_0 is the zeroth-order Hankel function of the first kind.
For an incident excitation $W^{\mathrm{i}}(\boldsymbol{r})$, the multiple scattering problem is solved by setting up a system of self-consistent equations, with the solution for the displacement $W(\boldsymbol{r})$ given by:

$$W(\boldsymbol{r}) = W^{\mathrm{i}}(\boldsymbol{r}) + \sum_\alpha \phi_\alpha G\left(\boldsymbol{r}-\boldsymbol{R}_\alpha\right) \tag{2}$$

with $\phi_\alpha = T_\alpha(\omega)\psi^{\mathrm{e}}\left(\boldsymbol{R}_\alpha\right)$, the so-called "external" field $\psi^{\mathrm{e}}\left(\boldsymbol{R}_\alpha\right)$ representing the incident field on scatterer α, and the coefficients $T_\alpha = \frac{t_\alpha}{1-it_\alpha/(8k^2)}$. One finally obtains the system of equations in matrix form $M\Phi = \Psi^i$, with the elements of the matrix given by $M_{\alpha\beta} = \delta_{\alpha\beta}t_\alpha^{-1} - G\left(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta\right)$ and the right-hand side is a vector containing the values of the incident field at the resonators' positions $\Psi_\alpha^i = W^{\mathrm{i}}(\boldsymbol{R}_\alpha)$.

The nonlinear eigenvalue problem consists in seeking solutions of the governing equations without excitation, hence setting $W^{\mathrm{i}} = 0$, one needs to find the eigenfrequencies ω_n and eigenvectors Φ_n satisfying $M(\omega_n)\Phi_n = \boldsymbol{0}$. There are various methods to solve this, which can be broadly separated into two families, contour integral techniques [6] or iterative methods [7]. Each type has its advantage and inconvenient, contour integral methods can compute all singularities within a given region of the complex plane but can be slow because they require computing integrals on a closed path accurately, whereas iterative methods are faster but require an initial guess. Here we use a variant of the latter, namely Newton's method with a generalized Rayleigh quotient iteration [7] to find the eigenvalues and eigenvectors.

Next we derive a modal expansion, according to Keldyšh theorem [8]:

$$M^{-1}(\omega) = \sum_n \frac{1}{\omega-\omega_n}\frac{\Phi_n\Phi_n}{\Phi_n\cdot M'(\omega_n)\Phi_n} + h(\omega) \tag{3}$$

with h a matrix valued analytic function representing the non-resonant background which is zero if M is a strictly proper rational function [6]. However, this expansion is not unique as the system is overcomplete. Neglecting the non-resonant term and following [9], we take a generalisation of the previous theorem and write the expansion for the solution:

$$\Phi = \sum_n \frac{f(\omega_n)}{f(\omega)}\frac{1}{\omega-\omega_n}\frac{\Phi_n\cdot\Psi^i}{\Phi_n\cdot M'(\omega_n)\Phi_n}\Phi_n = \sum_n b_n(\omega)\Phi_n \tag{4}$$

In the case of polynomial or rational eigenproblems, it has been shown in [9] that f is polynomial of maximum degree depending on the dispersive properties. The situation is more complicated here as we use a Green's function formalism involving transcendental functions. For the cases tested numerically it seems that $f = 1/k^3$ works best, and we will use this in the examples.

III. Example

The tools provided by the eigen-analysis enables rapid exploration of parametric studies involving intricate geometries of scatterers. We are particularly interested in graded line arrays of resonators, due to the intriguing properties arising from the excitation of the eigenmodes. By grading the array, we induce rainbow trapping, wherein the constituent frequencies of a source spatially separate, resulting in a band-gap cut-off frequency distributed at various positions along the array. This phenomenon, well-established in optics, has also been observed in acoustics, water waves and elasticity [10]. The chosen configuration consists of 10 resonators separated by a distance a, with resonant frequencies $\omega_{R\alpha}$ linearly distributed between ω_p and $0.8\omega_p$, ($\omega_p^2 = D/\rho h a^2$ being the fundamental frequency of the plate), with stiffnesses $k_{R\alpha} = k_p = \omega_p^2 m_p$ (with $m_p = \rho h a^2$) and masses $m_{R\alpha} = k_p/\omega_{R\alpha}^2$,

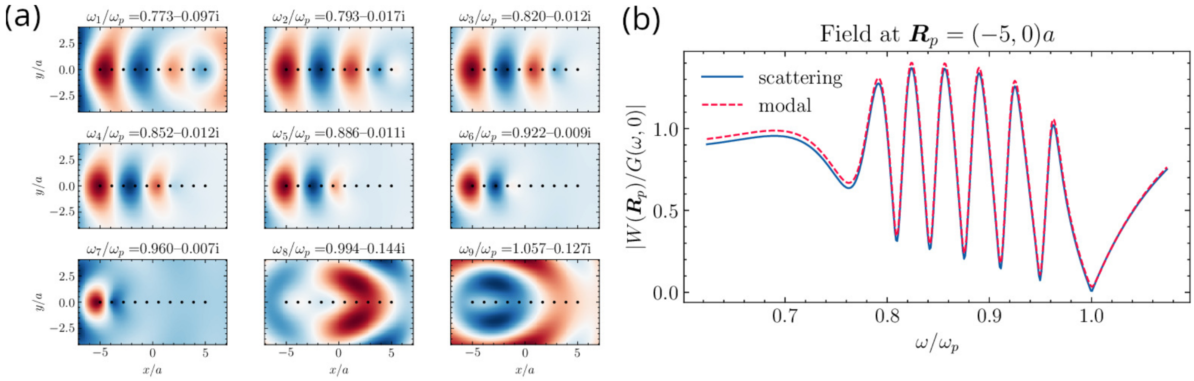

Fig. 1: Eigenmodes and modal expansion for a graded line array of resonators. (a): QNMs displacement field corresponding to eigenfrequencies with the lowest imaginary part. (b): Normalised displacement field norm excited and evaluated at the leftmost resonator computed with the scattering problem and modal expansion as a function of frequency.

the eigenmodes are displayed on Fig.(1a). Their excitation by a point source gives rise to a resonant interaction with multiple spectrally separated peaks in the total field (solid blue line on Fig. (1b)). The reconstruction with 16 QNMs allows a fast evaluation of the displacement field as a function of excitation frequency that is in excellent agreement with multiple scattering computations (dashed red line on Fig. (1b)).

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