IMPERIAL

Modal Analysis for Controlling Elastic Waves in Platonic Metamaterials

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09/09/2024

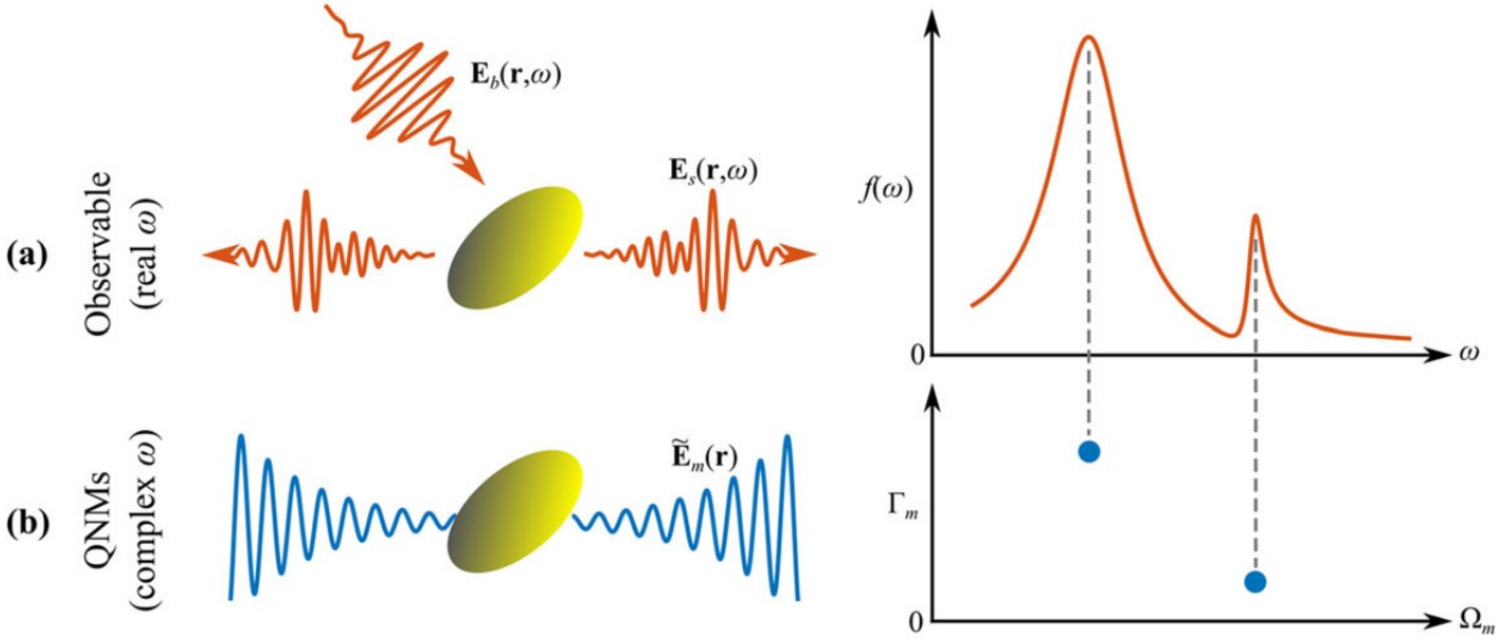
Metamaterials 2024, Chania, Greece

Introduction

Quasi Normal Modes (QNMs)

AKA:
- scattering resonances
- resonant states
- leaky modes
- quasi BICs

Well studied in electromagnetism



Lalanne et al. Laser & Photonics Reviews 12, 1700113 (2018).

Both et al. Semicond. Sci. Technol. 37, 013002 (2021).

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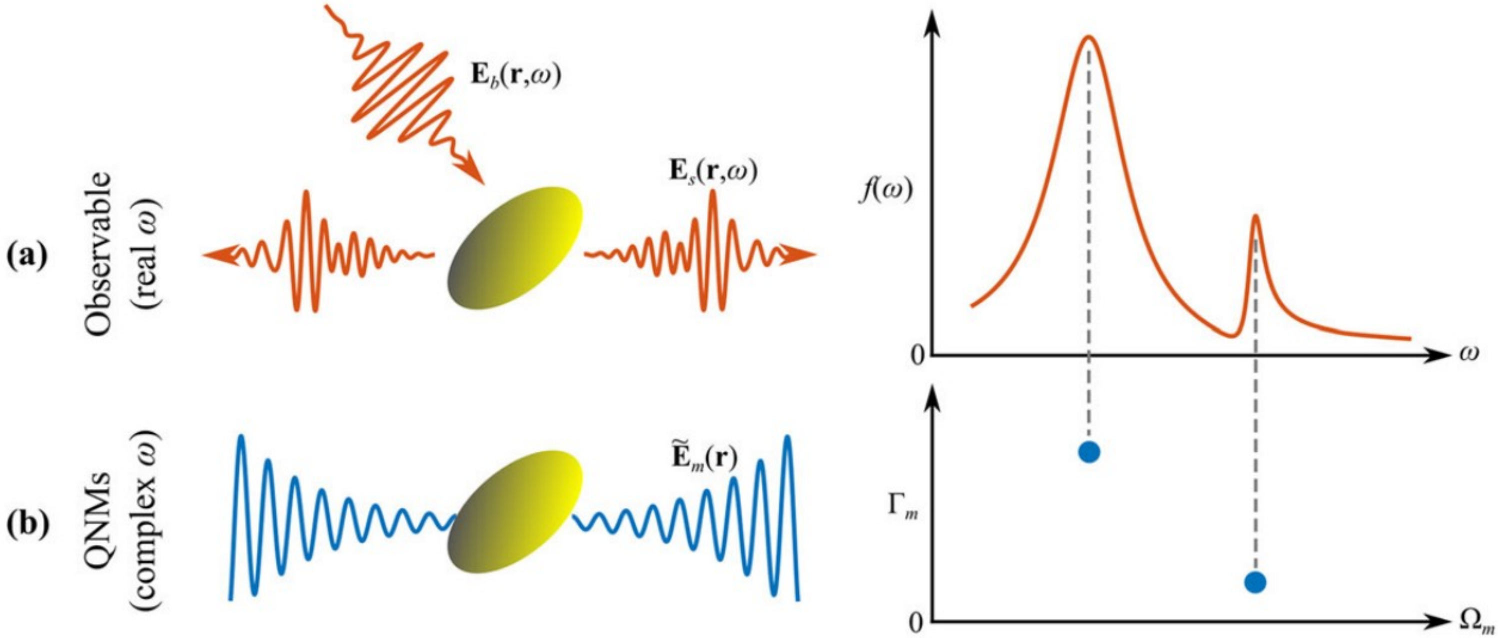

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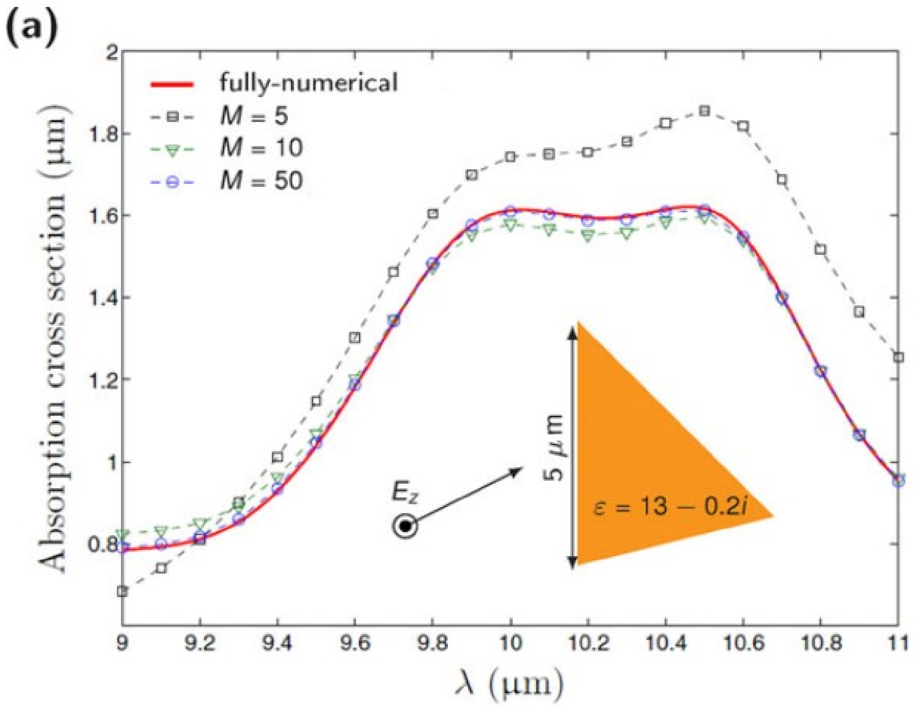

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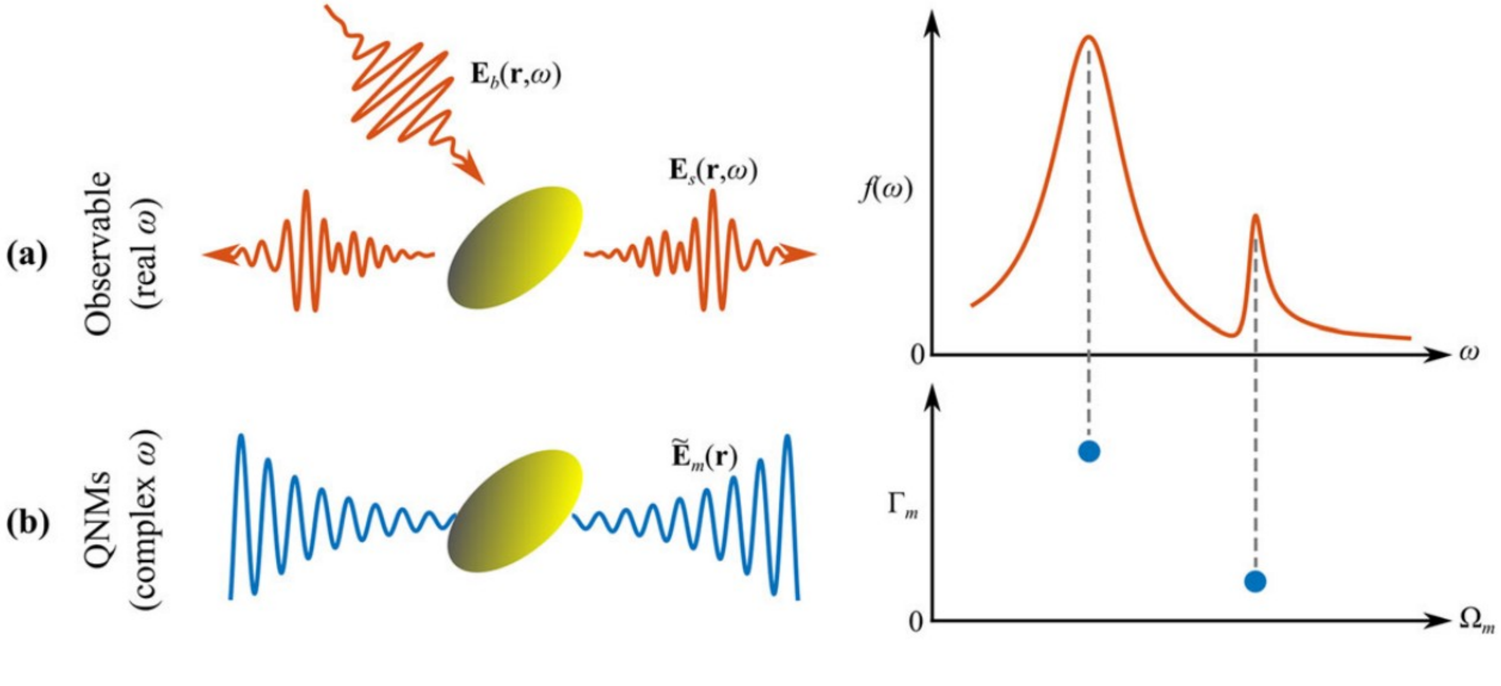

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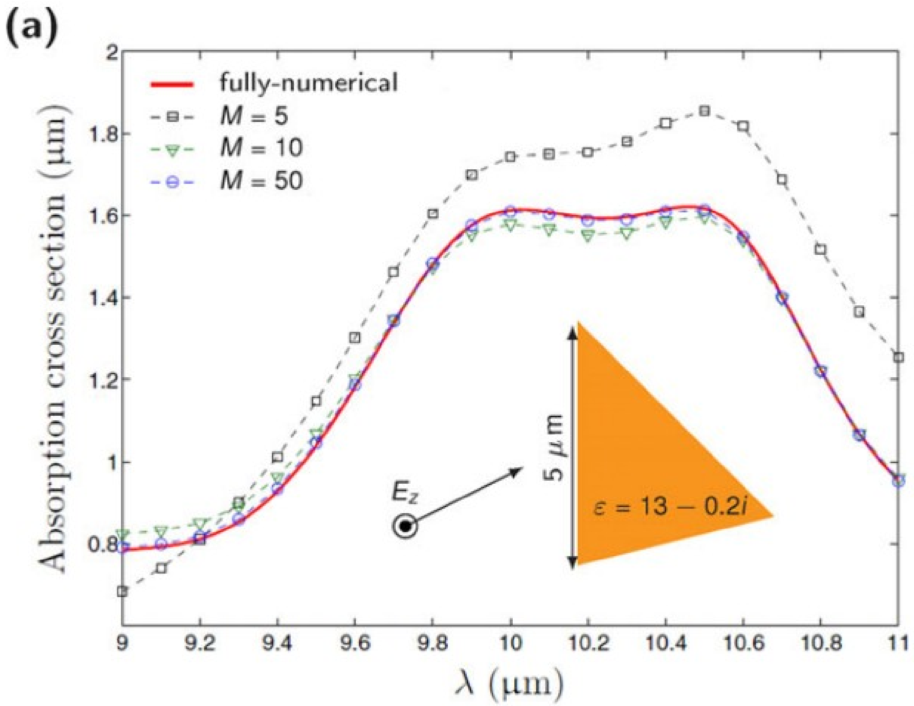

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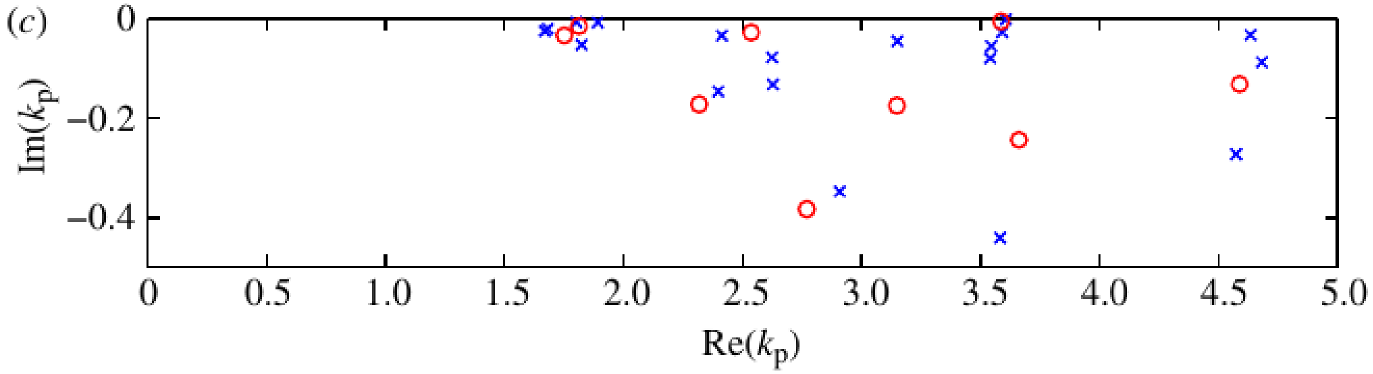
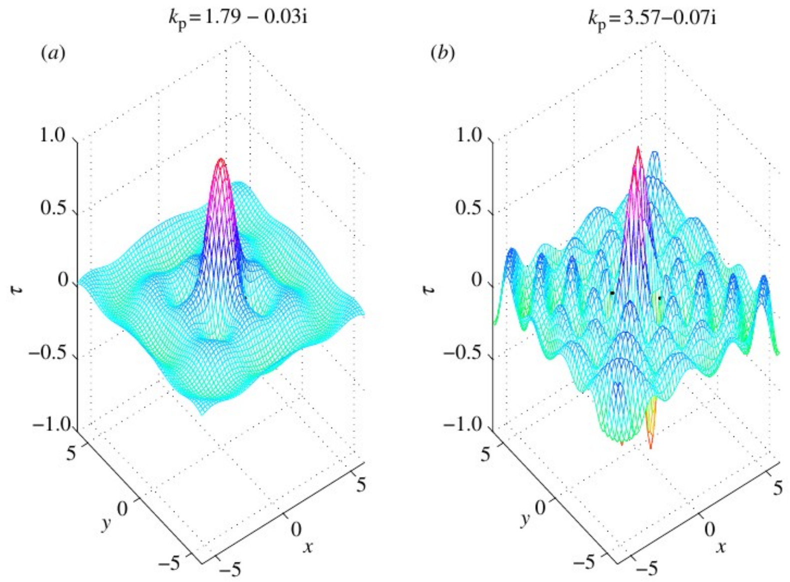
Extension to frequency dispersive materials

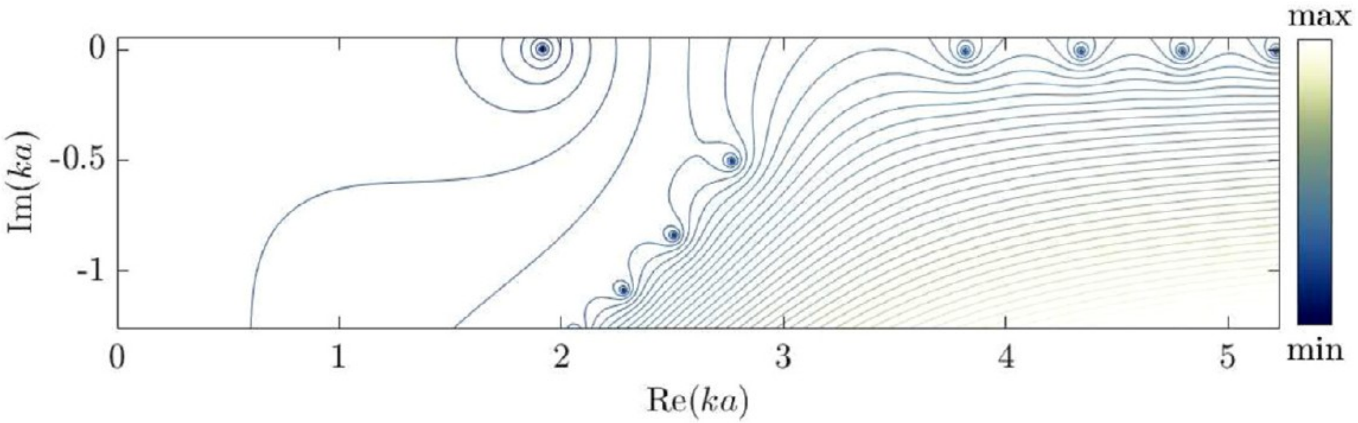
Zolla et al. Opt. Lett., OL 43, 5813–5816 (2018).

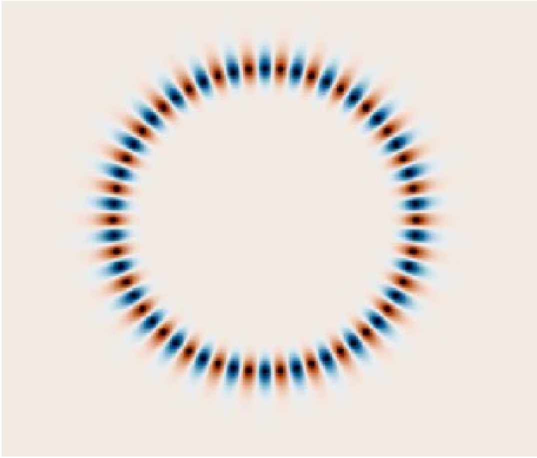
Quasi Normal Modes in elasticity





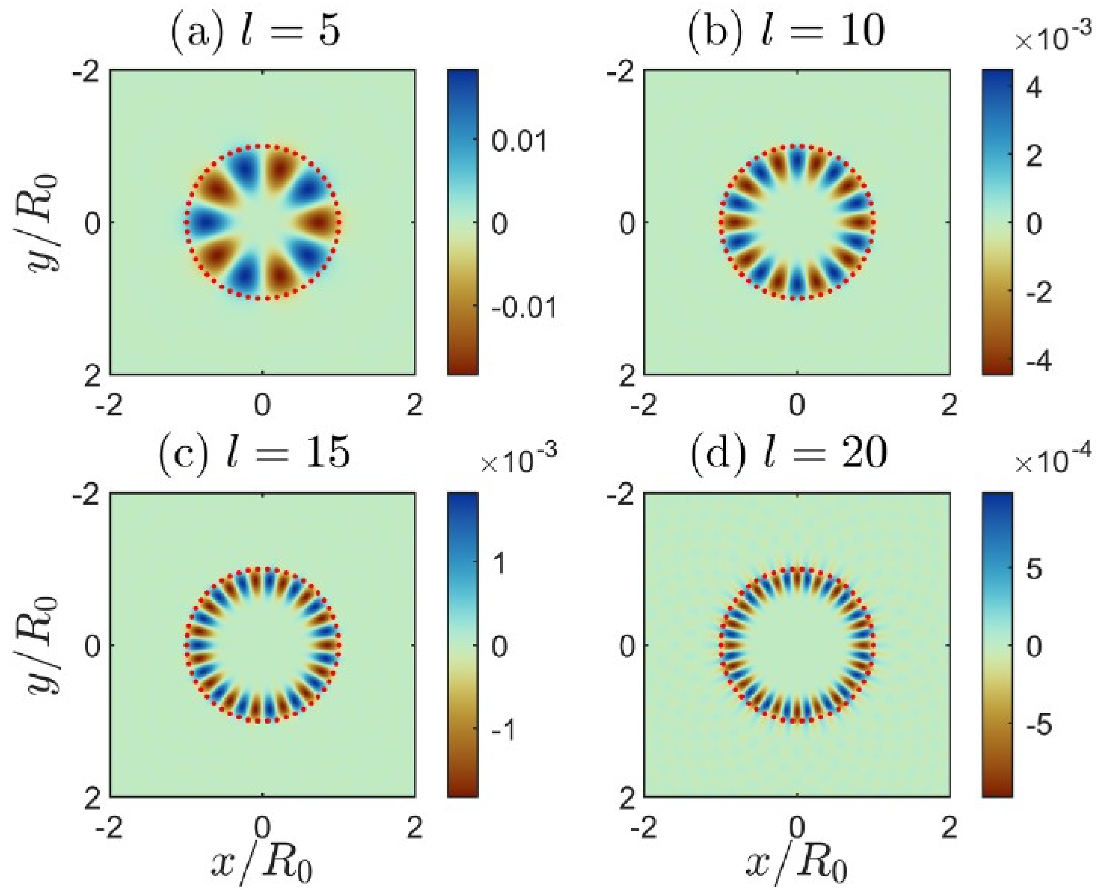
Meylan et al., Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 467, 3509–3529 (2011)



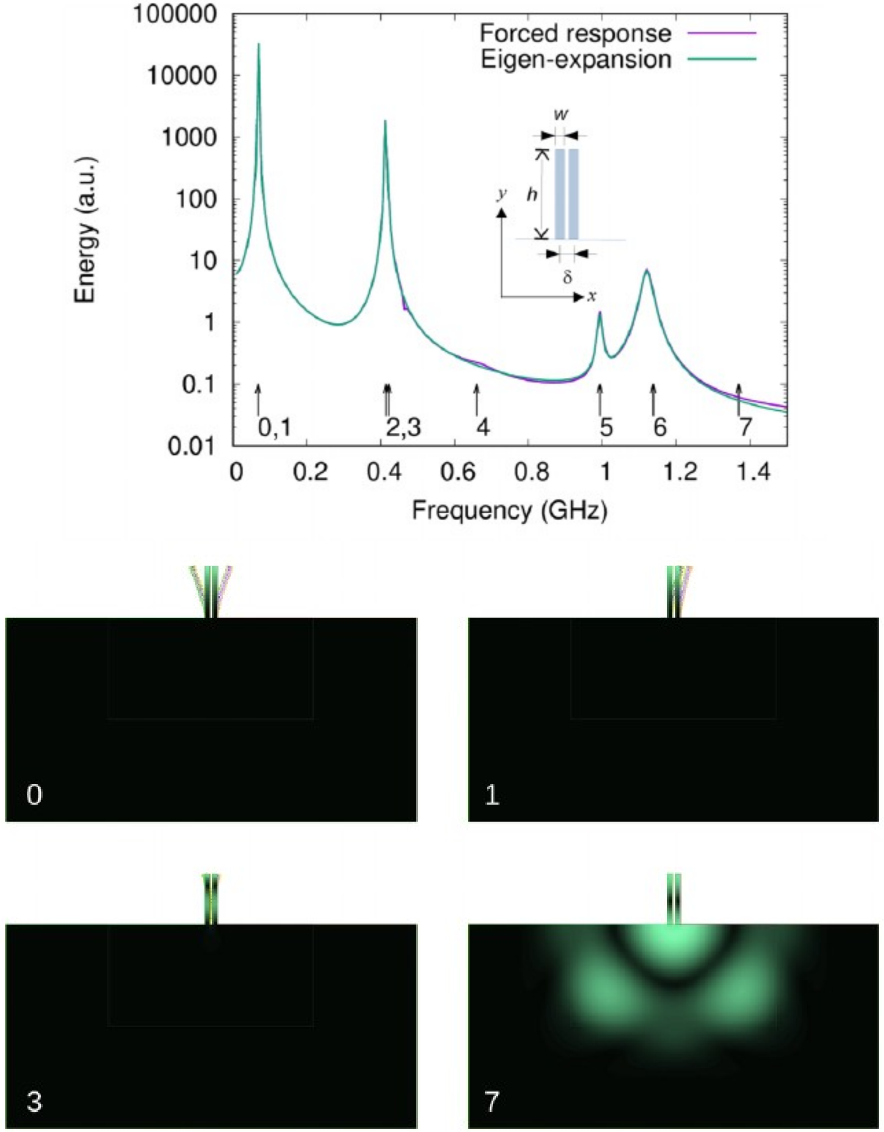


Putley, et al. Wave Motion 105, 102755 (2021).

Quasi Normal Modes in elasticity



Martí-Sabaté et al. Phys. Rev. Res. 5, 013131 (2023).

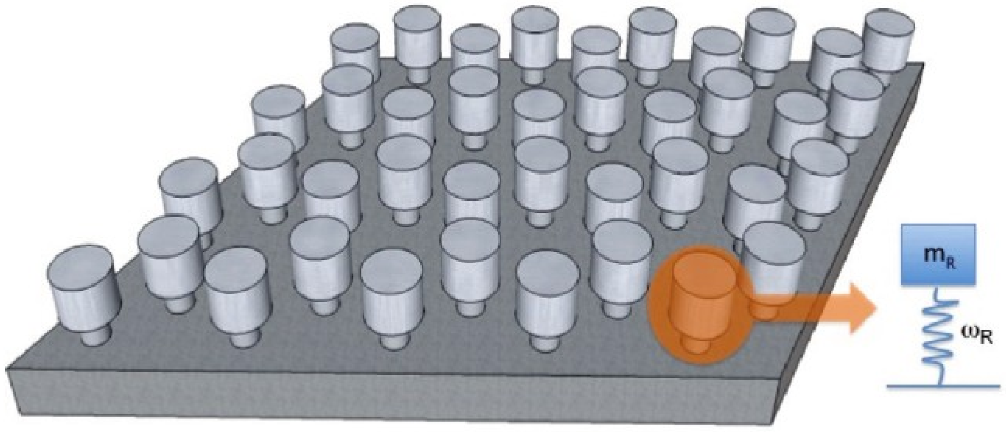


Laude et al. Phys. Rev. B 107, 144301 (2023).

Elastic waves on thin elastic plates: scattering and modal analysis

Equations of motion

Kirchoff Love theory for thin elastic plates



wavenumber $k^4 = \frac{\omega^2 h}{D}$

bending stiffness $D = \frac{Eh^3}{12(1-\nu^2)}$

Displacement

$$\mathcal{P}(\omega)W(\mathbf{r}) = \left(\boldsymbol{\nabla}^4 - k^4 - \mathcal{R}(\omega)\right)W(\mathbf{r}) = 0$$

Resonators contribution

$$\mathcal{R}(\omega)W(r) = \sum_\alpha t_\alpha(\omega)W(R_\alpha)\delta(r - R_\alpha).$$

Strength / impedance

$$t_\alpha(\omega) = \frac{m_{R\alpha}}{D}\frac{\omega_{R\alpha}^2\omega^2}{\omega_{R\alpha}^2 - \omega^2}$$

Torrent et al. Phys. Rev. B 87, 115143 (2013).

Multiple scattering

Green's funtion for bare plate

$$G(\mathbf{r}) = \frac{i}{8k^2}\left[H_0(kr) - H_0(ikr)\right]$$

Displacement

$$W(\mathbf{r}) = W^i(\mathbf{r}) + \sum_\alpha \phi_\alpha G\left(\mathbf{r} - \mathbf{R}_\alpha\right)$$

Linear system

$$M\Phi = \Psi^i$$

$$M_{\alpha\beta} = \delta_{\alpha\beta} t_\alpha^{-1} - G\left(\mathbf{R}_\alpha - \mathbf{R}_\beta\right)$$

$$\Psi^i_\alpha = W^i(\mathbf{R}_\alpha)$$

Torrent et al. Phys. Rev. B 87, 115143 (2013).

Eigenvalue problem

No sources!

$$M(\omega_n)\Phi_n = 0$$

eigenfrequencies   eigenmodes

Nonlinear eigenvalue problem

How to solve this?

W.-J. Beyn, An integral method for solving nonlinear eigenvalue problems, Linear Algebra and its Applications , 436, 3839 (2012).
H. Chen, On locating the zeros and poles of a meromorphic function, Journal of Computational and Applied Mathematics 402, 113796 (2022).
M. Van Barel and P. Kravanja, Nonlinear eigenvalue problems and contour integrals, Journal of Computational and Applied Mathematics 292, 526 (2016).
S. Güttel and F. Tisseur, The nonlinear eigenvalue problem, Acta Numerica 26, 1 (2017).

Eigenvalue problem

Contour integrals with iterative refinement

$$M(\omega_n)\Phi_n = 0$$

eigenfrequencies   eigenmodes

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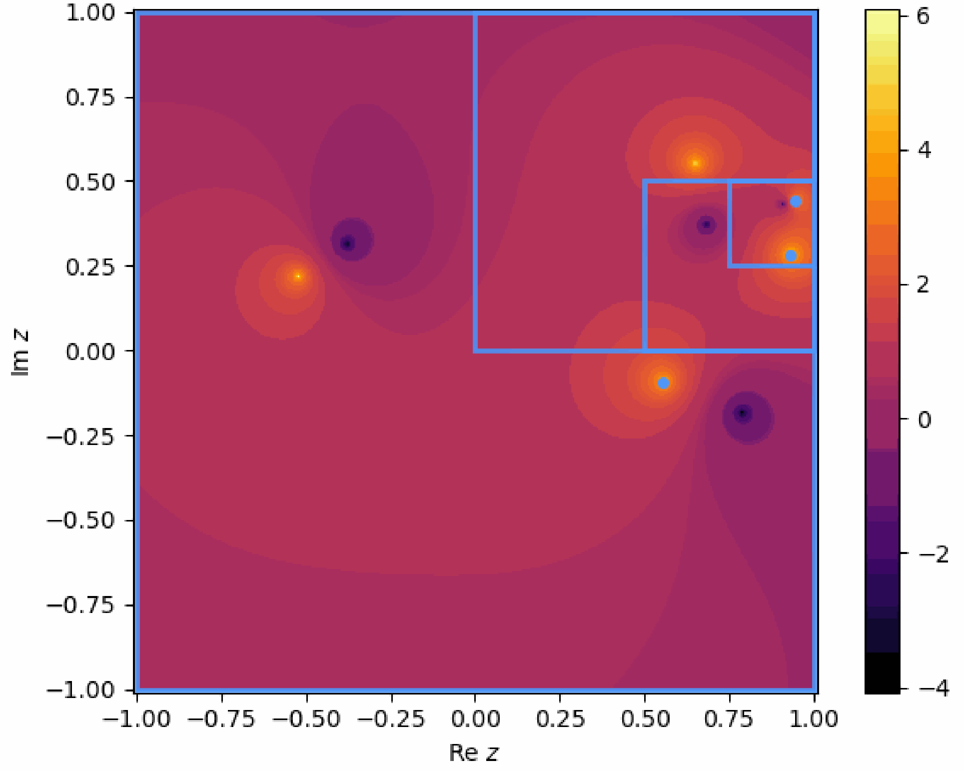

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- Contour integral techniques: *need accurate computation of integrals along a closed path in the complex plane*

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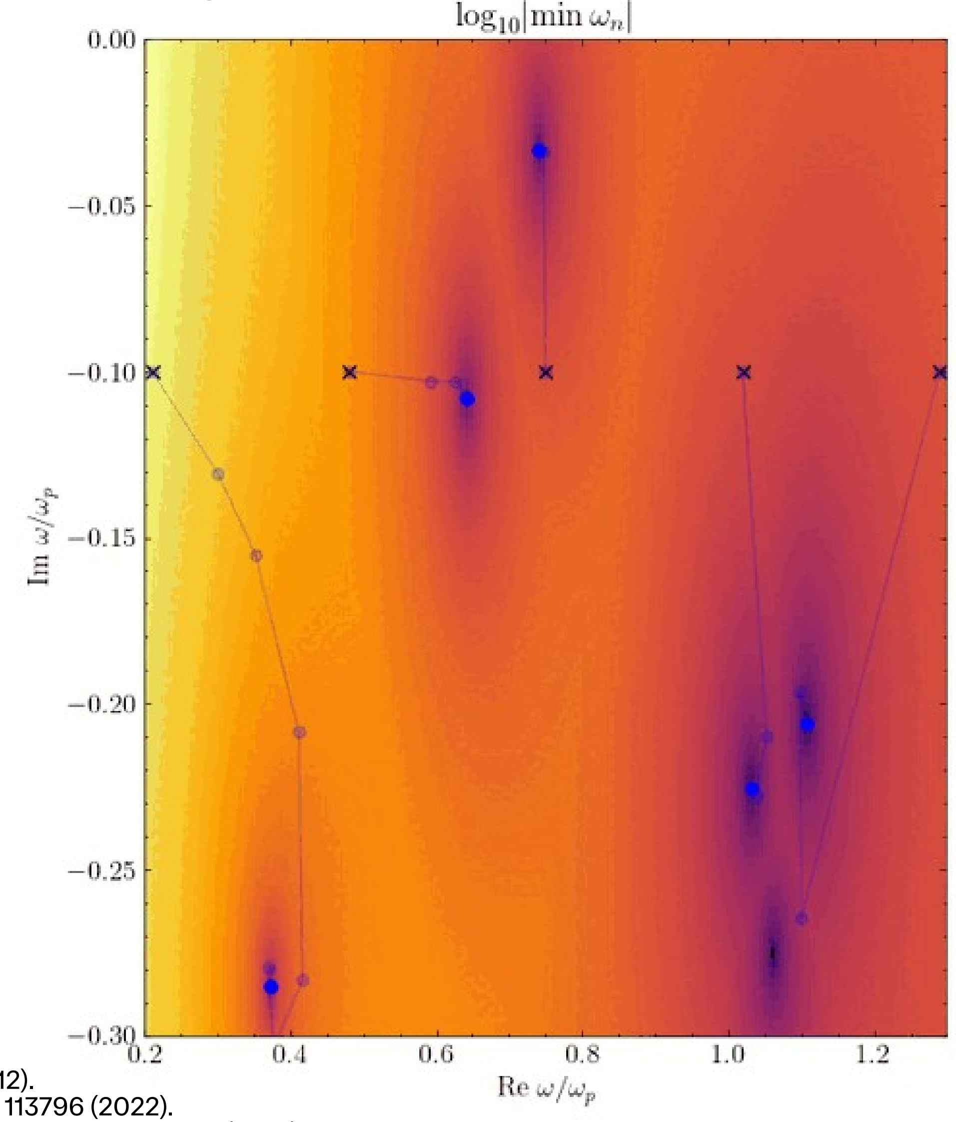
- Contour integral techniques: *need accurate computation of integrals along a closed path in the complex plane*
- Iterative methods (Rayleigh quotient): *need starting guess*

Rayleigh quotient



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Eigenvalue problem

Rayleigh quotient *Iterative grid search*

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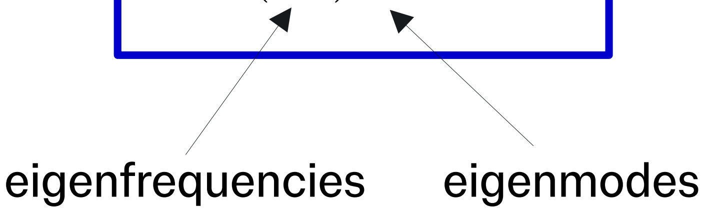
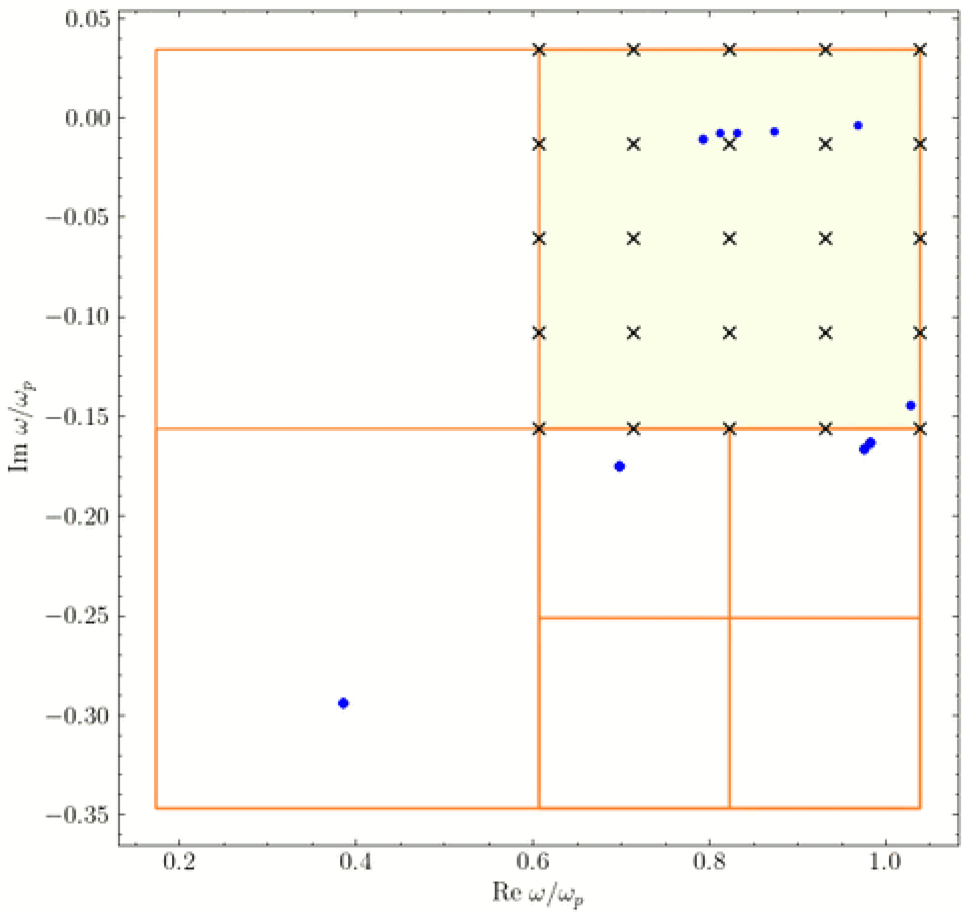
$$M(\omega_n)\Phi_n = 0$$





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Eigenvalue problem

Rayleigh quotient *Local maxima estimate*

No sources!

$$M(\omega_n)\Phi_n = 0$$

eigenfrequencies eigenmodes

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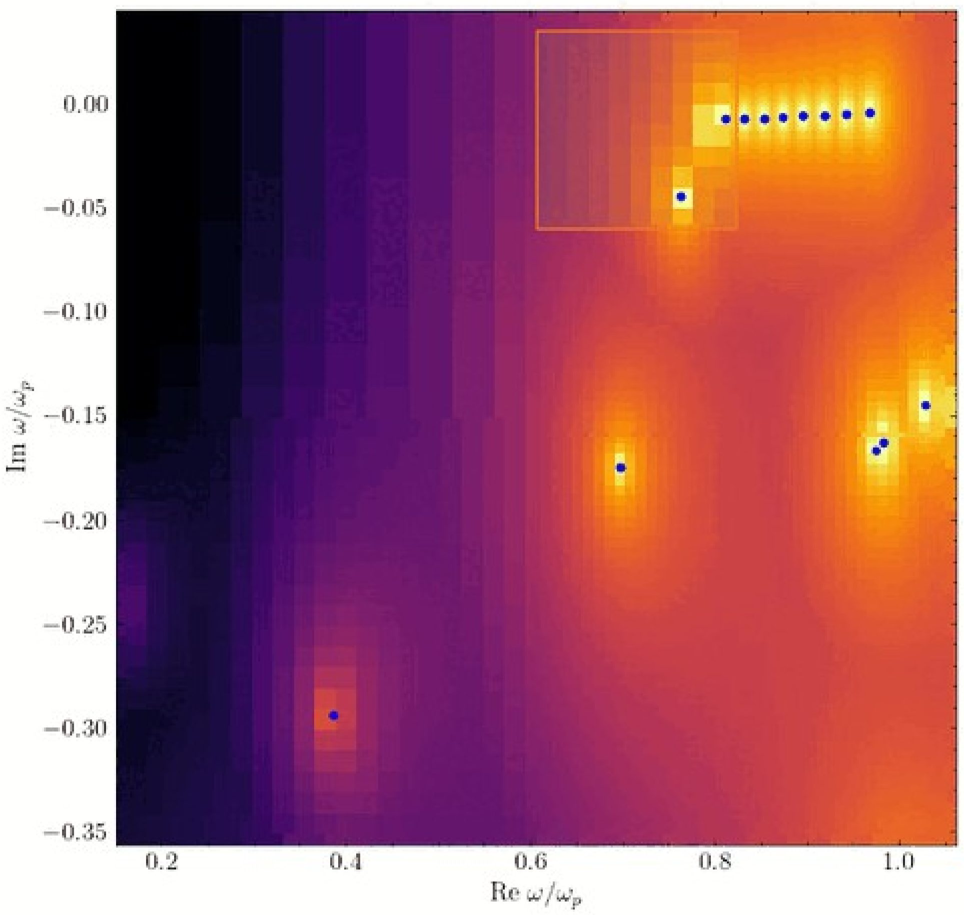

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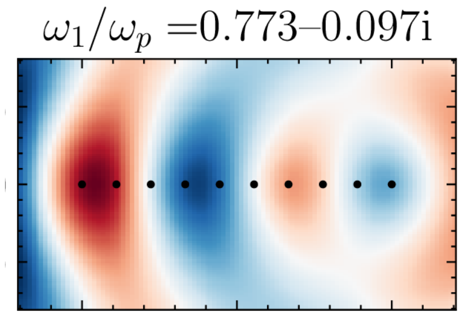
Example: graded line array

Rainbow trapping



Neglecting coupling between resonators

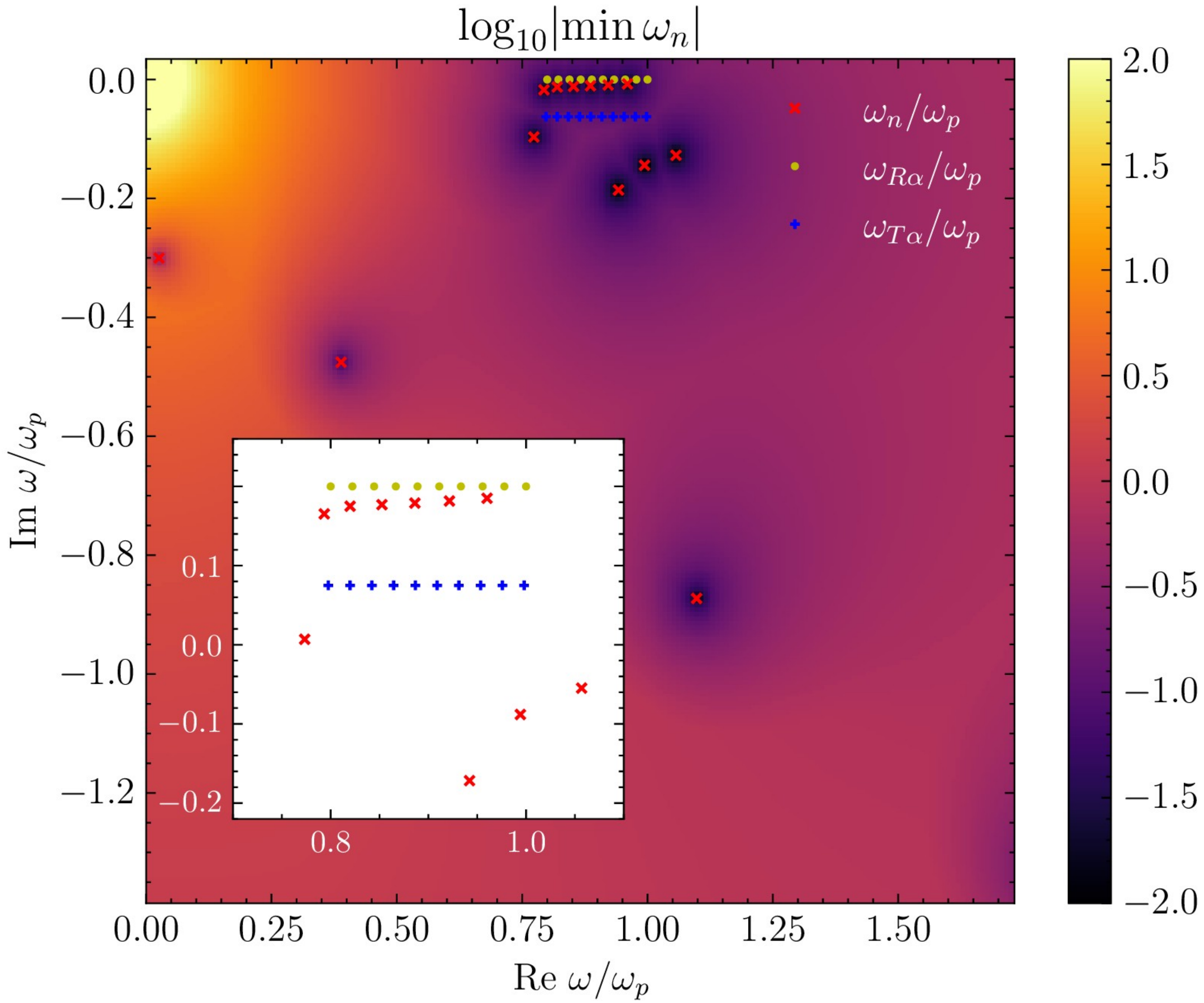
$$\omega^{\pm}_{T\alpha}/\omega_{R\alpha} = \pm\sqrt{1-q_\alpha^2} - iq_\alpha$$

with $q_\alpha = \frac{1}{16}\sqrt{\frac{k_{R\alpha} m_{R\alpha}}{\rho h D}} = \frac{k_{R\alpha} a}{16D}\frac{\omega_p}{\omega_{R\alpha}}$

if $q_\alpha \leq 1$, we have $\omega^{\pm}_{T\alpha}/\omega_{R\alpha} = \pm e^{i\theta_\alpha}$

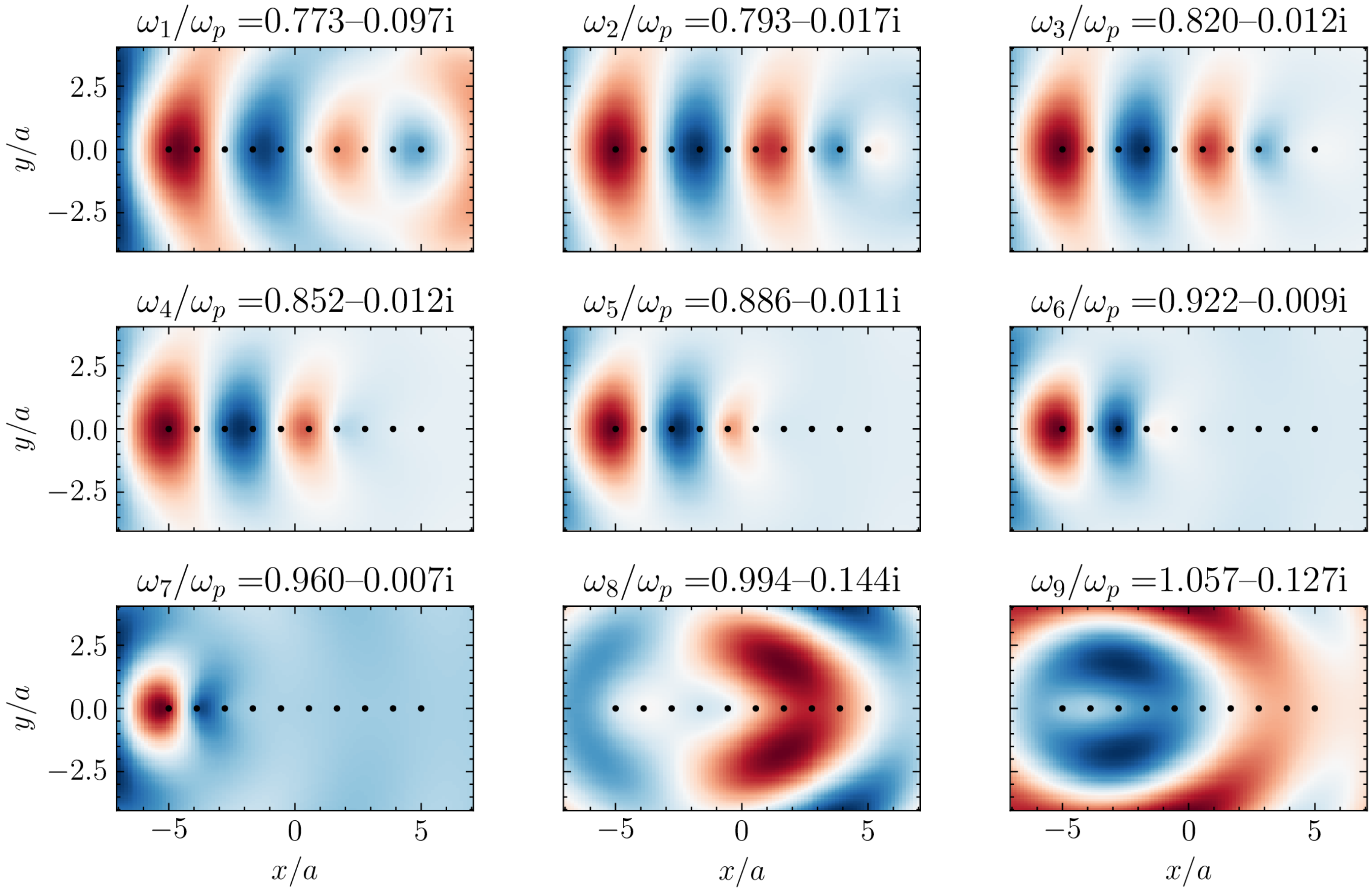
where $\theta_\alpha = -\arcsin(q_\alpha)$

→ Rotation in the complex plane



Graded line array

Eigenmodes



Quasi Normal modes expansion

Mode expansion

Keldysh theorem

$$M^{-1}(\omega) = \sum_n \frac{1}{\omega - \omega_n} \frac{\Phi_n \Phi_n^T}{\Phi_n \cdot M'(\omega_n)\Phi_n} + h(\omega)$$

M. V. Keldysh, On some cases of degeneration of an equation of elliptic type on the boundary of a domain, Dokl. Akad. Nauk SSSR 77, 181 (1951)
M. D. Truong, A. Nicolet, G. Demésy, and F. Zolla, Continuous family of exact Dispersive Quasi-Normal Modal (DQNM) expansions for dispersive photonic structures, Optics Express 28, 29016 (2020)

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Overcomplete basis...

arbitrary function u, neglecting h

$$\Phi = \sum_n \frac{u(\omega_n)}{u(\omega)} \frac{1}{\omega - \omega_n} \frac{\Phi_n \cdot \Psi^i}{\Phi_n \cdot M'(\omega_n)\Phi_n} \Phi_n = \sum_n b_n(\omega)\Phi_n$$

For the cases tested numerically it seems that $u = 1/k^3 = 1/\omega^{3/2}$ works best.

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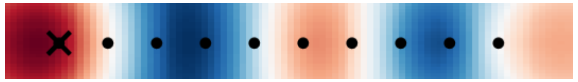
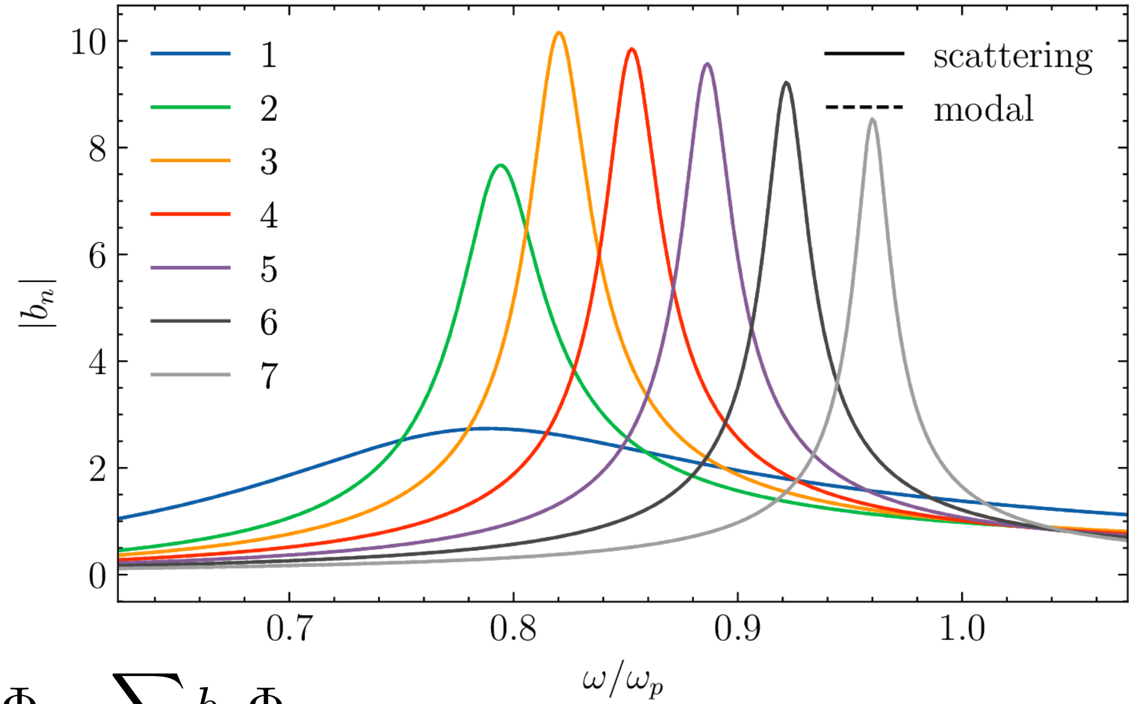
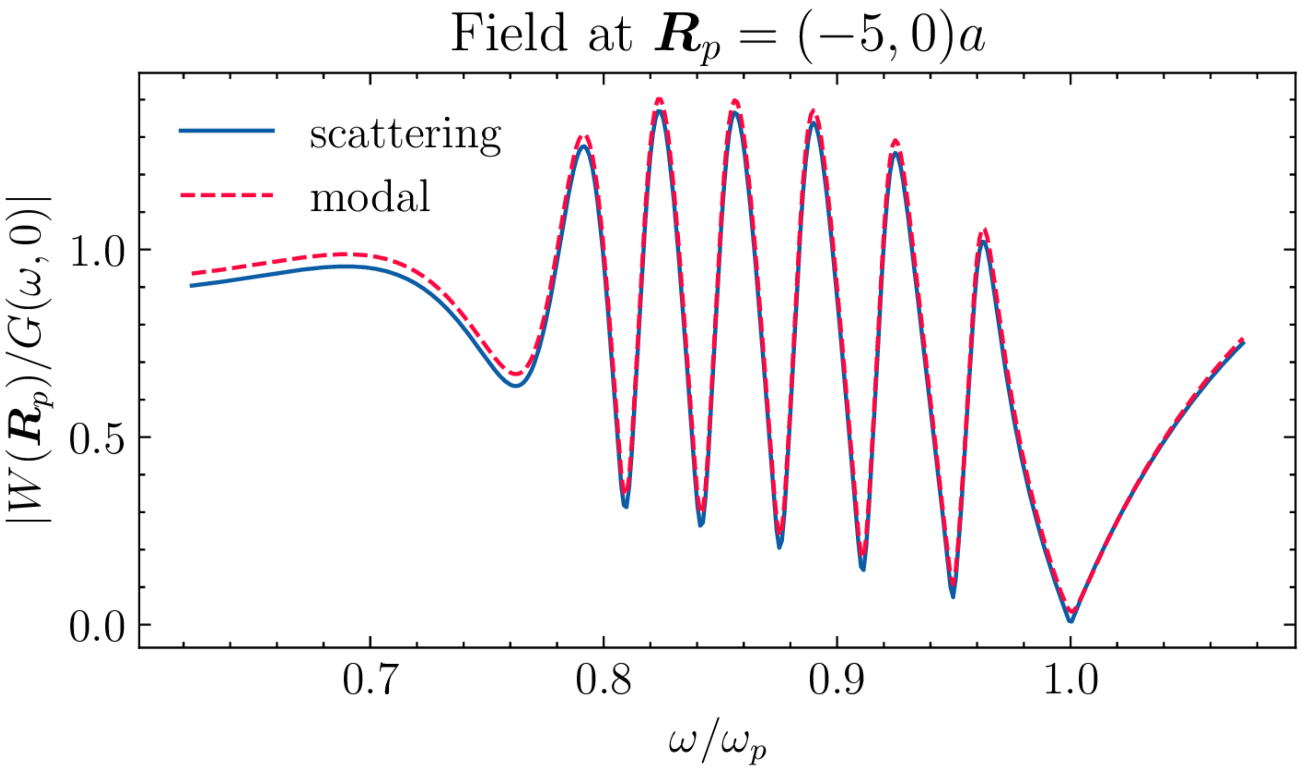


Excitation coefficients



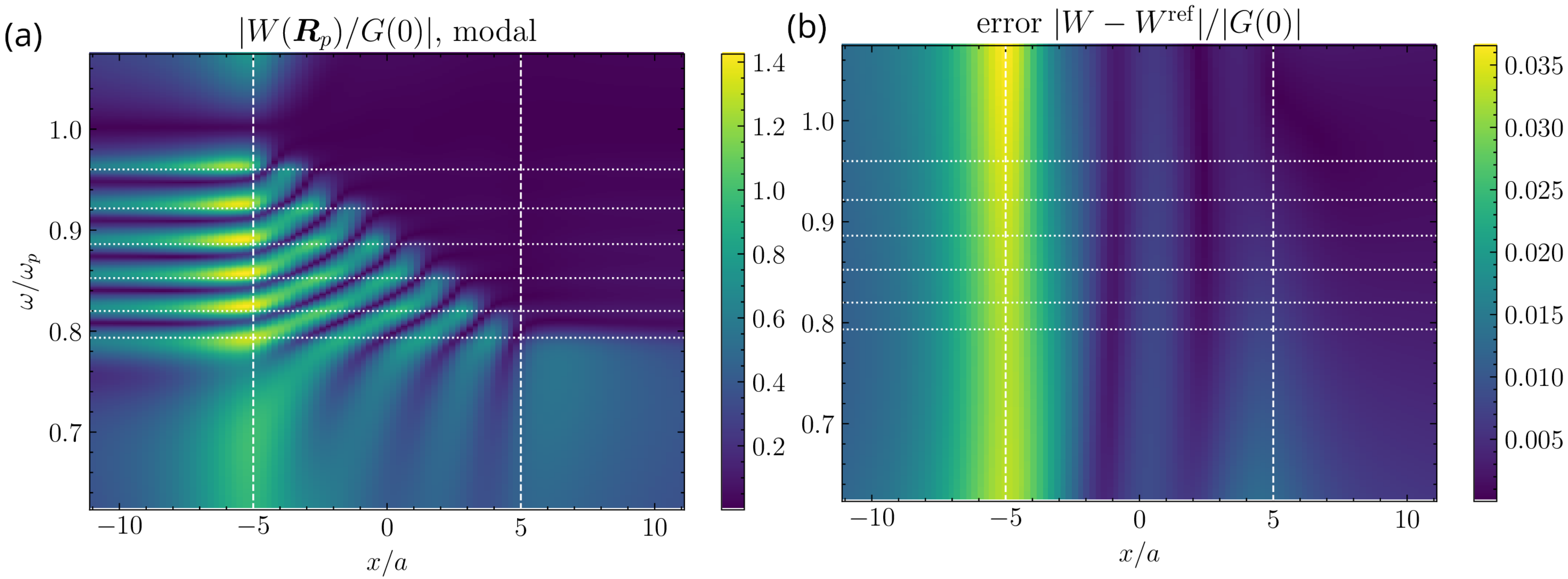
$$\Phi = \sum_n b_n \Phi_n$$

Field reconstruction

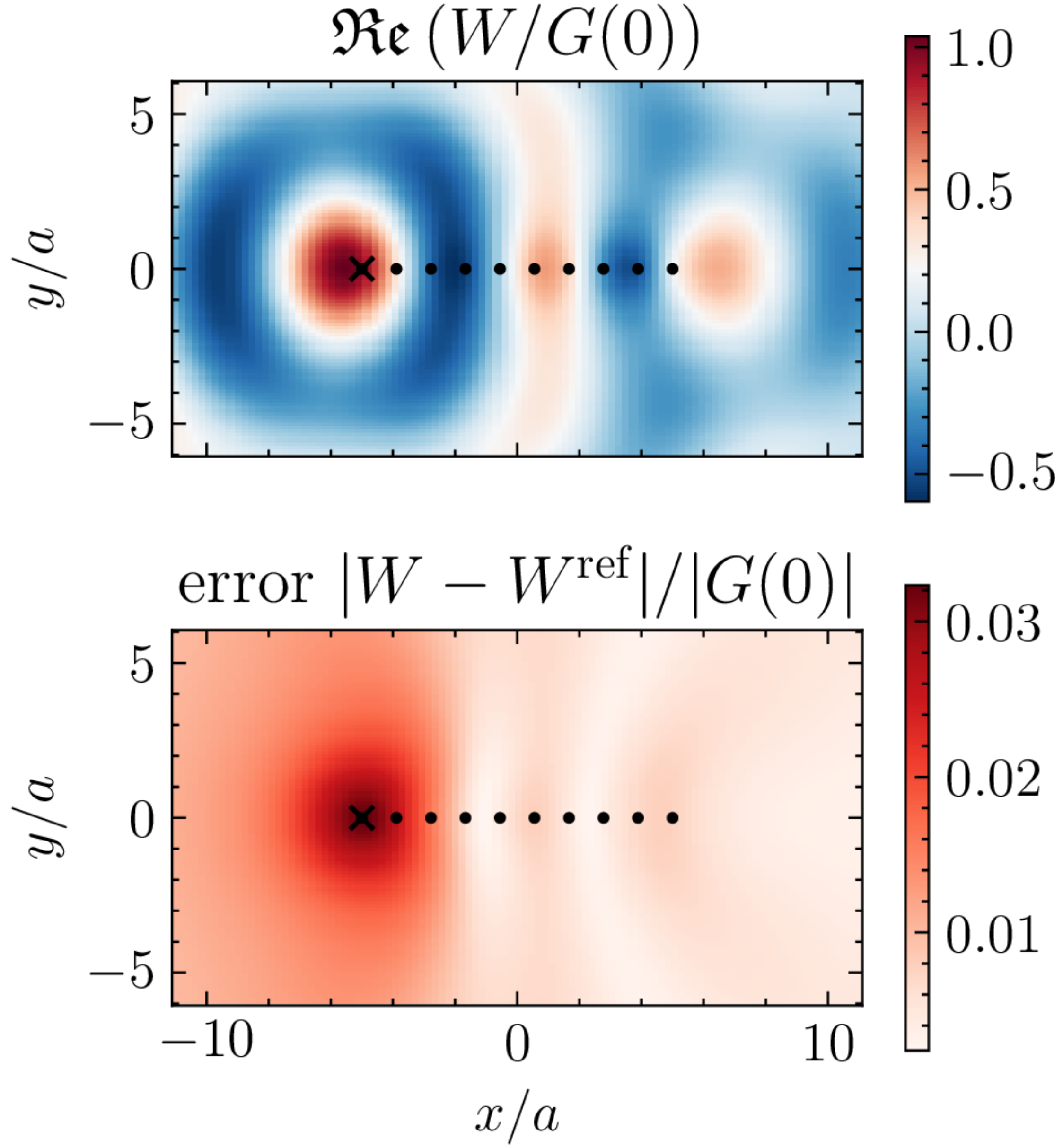


Graded line array

Displacement spectrum along the array



Graded line array
Displacement field



Green's function and LDOS
Modal expansion

Assuming the modes are normalized such that $\Phi_n M'(\omega_n)\Phi_n = 1$

Green's function

$$g(\omega,\mathbf{r},\mathbf{r}') = G(\omega,\mathbf{r}-\mathbf{r}') + \sum_n \frac{u(\omega_n)}{u(\omega)} \frac{1}{\omega-\omega_n} \sum_{\alpha,\beta} \Phi_{n,\alpha}\Phi_{n,\beta} G(\omega,\mathbf{r}-\mathbf{R}_\alpha) G(\omega,\mathbf{r}'-\mathbf{R}_\beta)$$

Local density of states

$$\begin{aligned}\mathcal{L}(\omega,\mathbf{r}) &= \frac{4k^3}{\pi}\mathrm{Im}\left[g(\omega,\mathbf{r},\mathbf{r})\right] \\ &= \mathcal{L}_0(\omega) + \frac{4k^3}{\pi}\sum_n \mathrm{Im}\left[\frac{u(\omega_n)}{u(\omega)} \frac{1}{\omega-\omega_n} \sum_{\alpha,\beta} \Phi_{n,\alpha}\Phi_{n,\beta} G(\omega,\mathbf{r}-\mathbf{R}_\alpha) G(\omega,\mathbf{r}-\mathbf{R}_\beta)\right]\end{aligned}$$

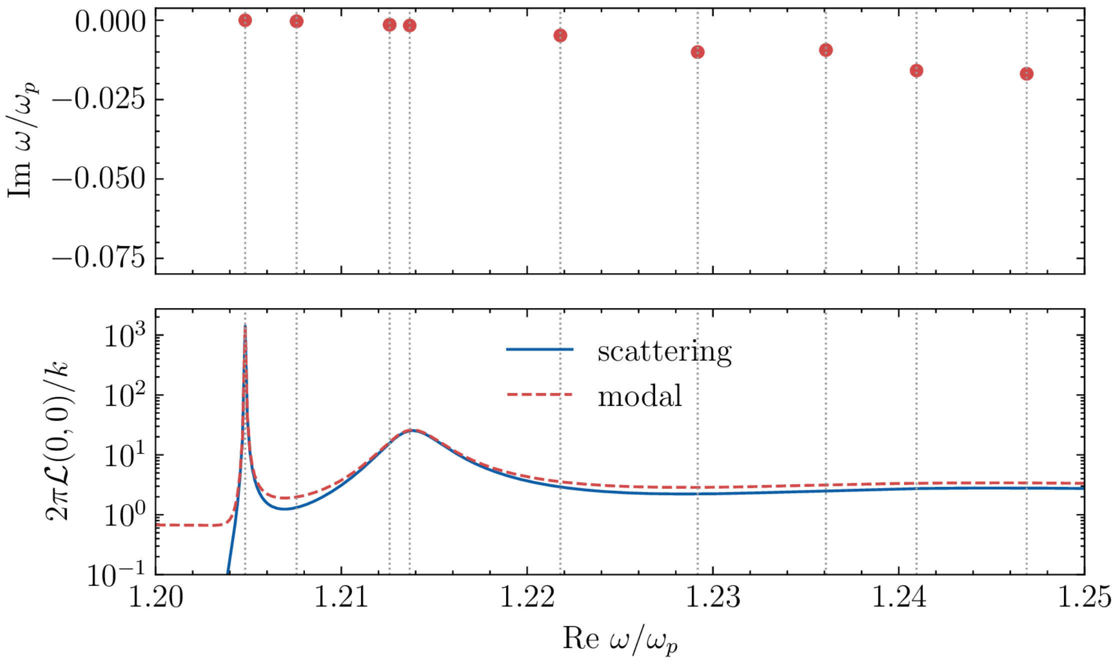
LDOS of the bare plate $\mathcal{L}_0(\omega) = k/2\pi$

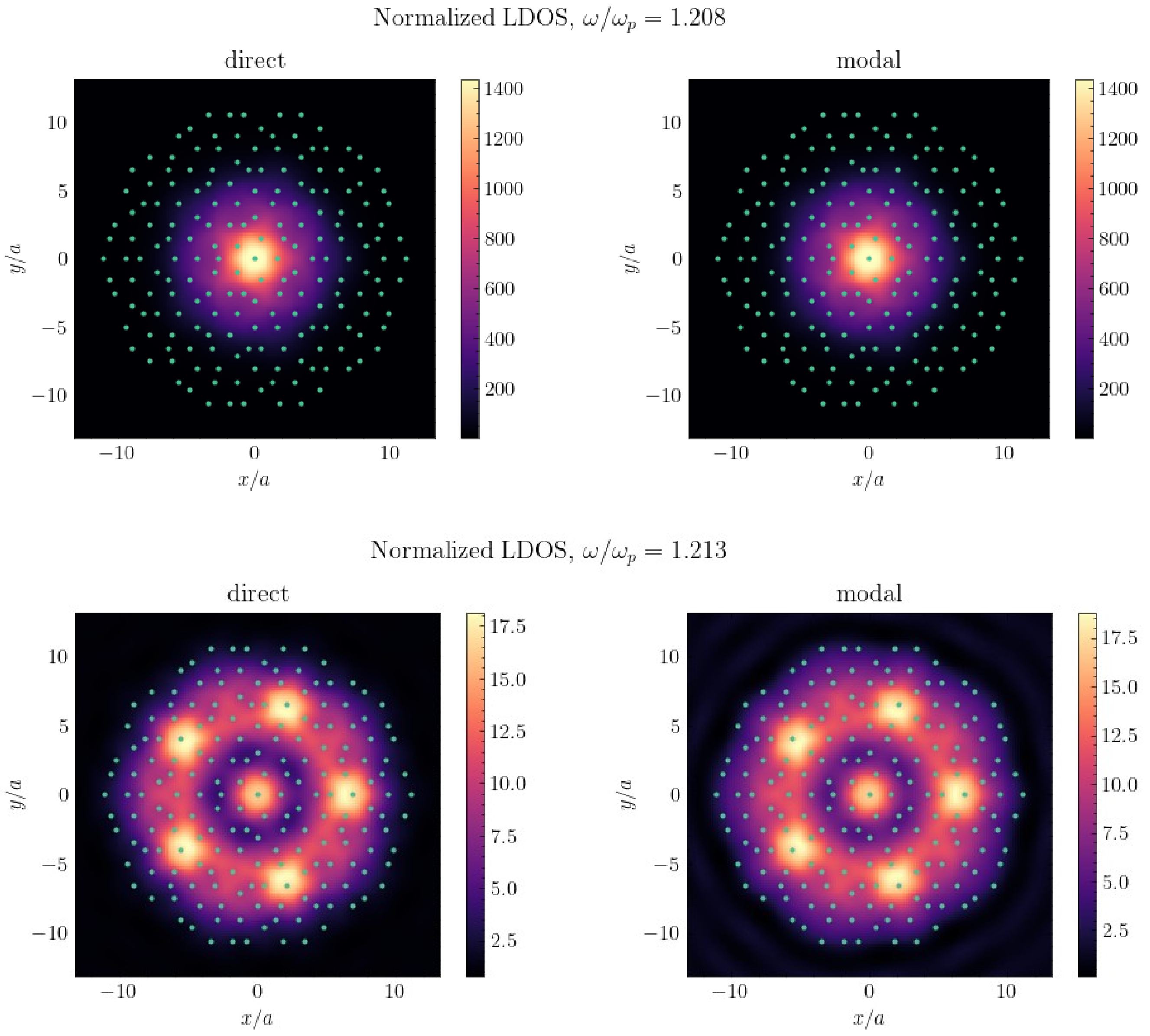
M. J. A. Smith, M. H. Meylan, and R. C. McPhedran, Density of States for Platonic Crystals and Clusters, SIAM Journal on Applied Mathematics 74, 1551 (2014)

Quasiperiodic cluster

191 resonators on a Penrose lattice

Normalized LDOS at the center





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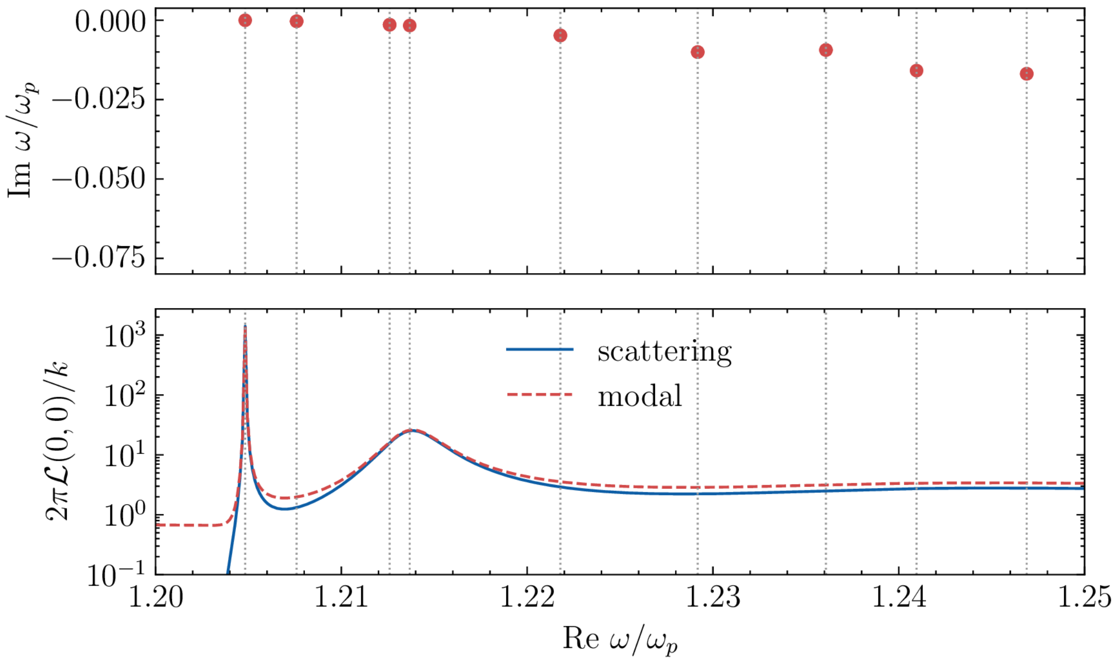

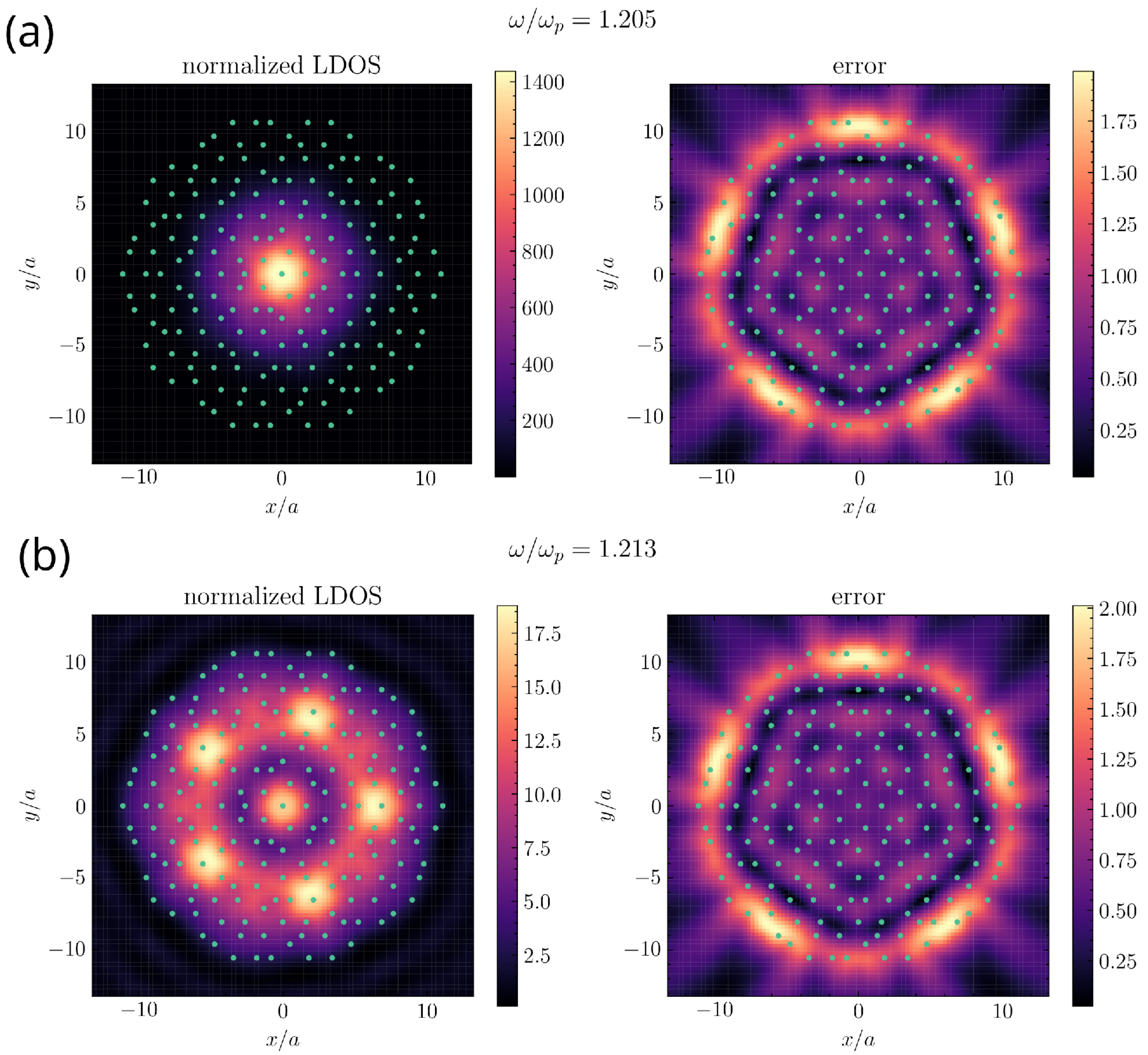

Excitation engineering

Orthogonality

The eigenvectors can be chosen orthogonal with respect to the generalised scalar product

$$\langle \Phi_m, \Phi_n \rangle_M := \begin{cases} \Phi_n \cdot \frac{M(\omega_m) - M(\omega_n)}{\omega_m - \omega_n} \Phi_m, & \text{if } \omega_m \neq \omega_n \\ \Phi_m \cdot M'(\omega_m) \Phi_m, & \text{if } \omega_m = \omega_n \end{cases}$$

Resonances engineering
Killing a mode

$$\Phi = \sum_n b_n \Phi_n$$

Choose

$$\Psi^i = \sum_{n \neq n_0} a_n \frac{M(\omega_{n_0}) - M(\omega_n)}{\omega_{n_0} - \omega_n} \Phi_n$$

for arbitrary complex valued a_n.
By construction $b_{n_0} \sim \Phi_{n_0} \cdot \Psi^i = 0$.
Incident field $W^i(\mathbf{r}) = \sum_n p_n W_n^i(\mathbf{r})$ such that:

$$W^i(\mathbf{R}_\alpha) = \sum_n p_n W_n^i(\mathbf{R}_\alpha) = \Psi_\alpha^i$$

for $\alpha = 1...N$, and this linear system is inverted to find p_n.

Resonances engineering
Killing a mode

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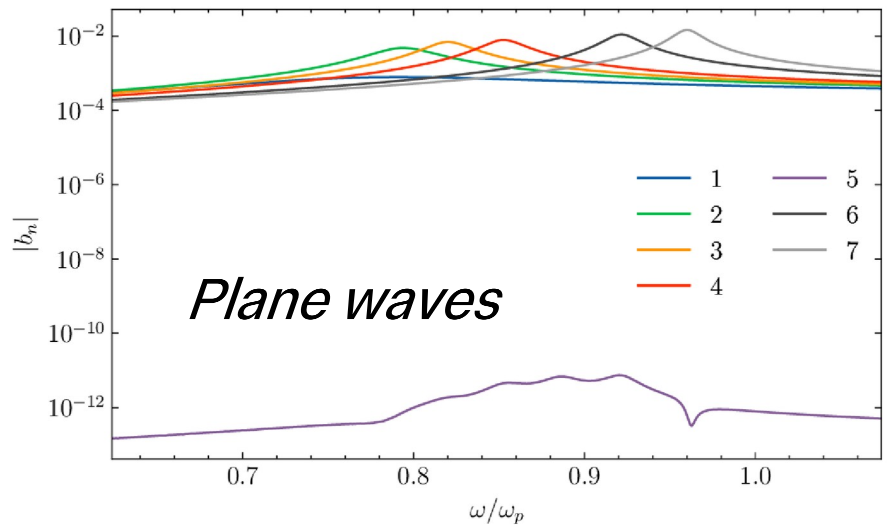

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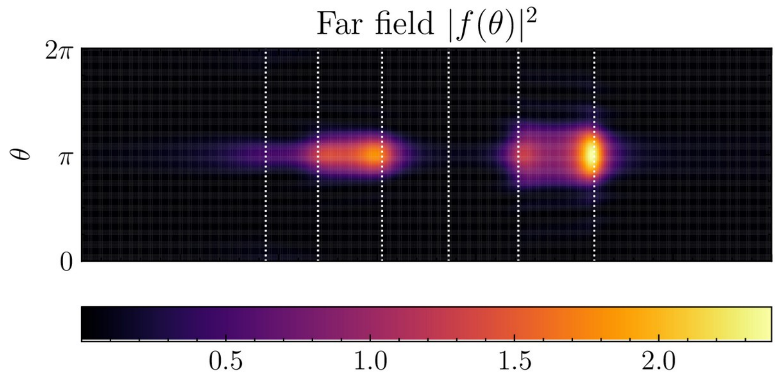

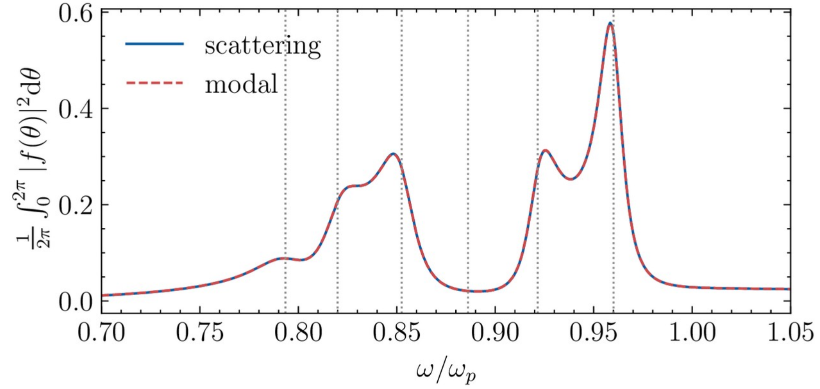

Example
Suppression of mode 5 contribution by designing an incident field as a linear combination of plane waves with angles evenly distributed between 0 and 2π.

Resonances engineering
Killing all modes except one

$$\Phi = \sum_n b_n \Phi_n$$

K_{n_0} the $N \times (N-1)$ rectangular matrix with eigenvectors Φ_n as columns apart from Φ_{n_0}.
By taking Ψ^i in the null space of K_{n_0}, we will have by definition $b_n \sim \Phi_n \cdot \Psi^i = 0$ for all $n \neq n_0$.

Resonances engineering

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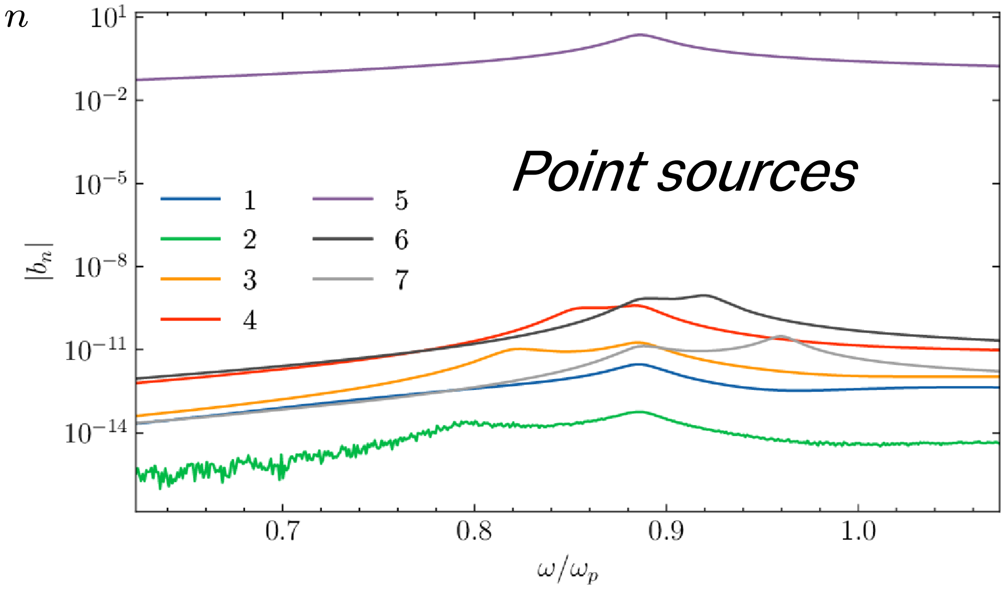

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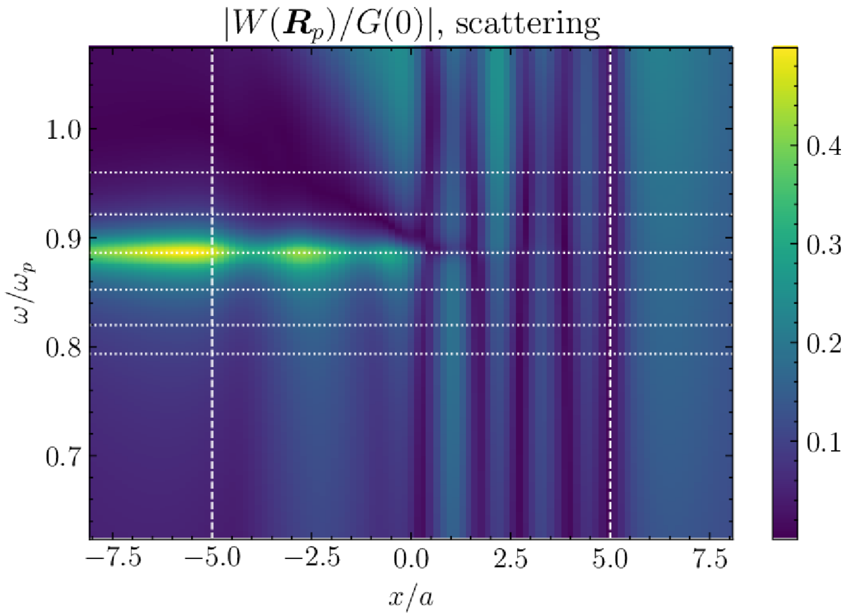
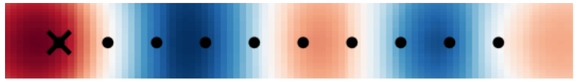


Example

A single eigenmode 5 is excited by designing a linear combination of 10 point sources located between the resonators at $x_{s,n} = x_{r,n} + a/2$



Optimization

Resonances engineering
Sensitivity analysis

$$\frac{\partial \omega_n}{\partial p} = -\Phi_n^T \frac{\partial M}{\partial p}(\omega_n)\Phi_n$$

Analytical formula

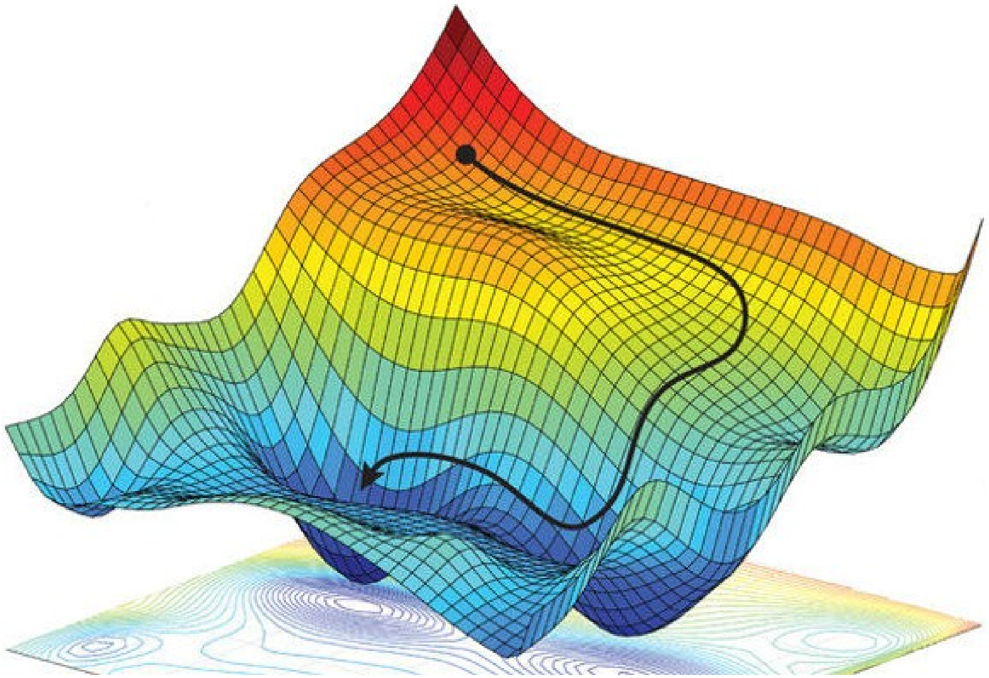
Resonators parameter:
position, mass, stiffness

Gradient of the objective functional $\mathcal{G}(p)$ to be minimized:

$$\frac{\partial \mathcal{G}}{\partial p} = \sum_n \frac{\partial \mathcal{G}}{\partial \omega_n}\frac{\partial \omega_n}{\partial p}.$$

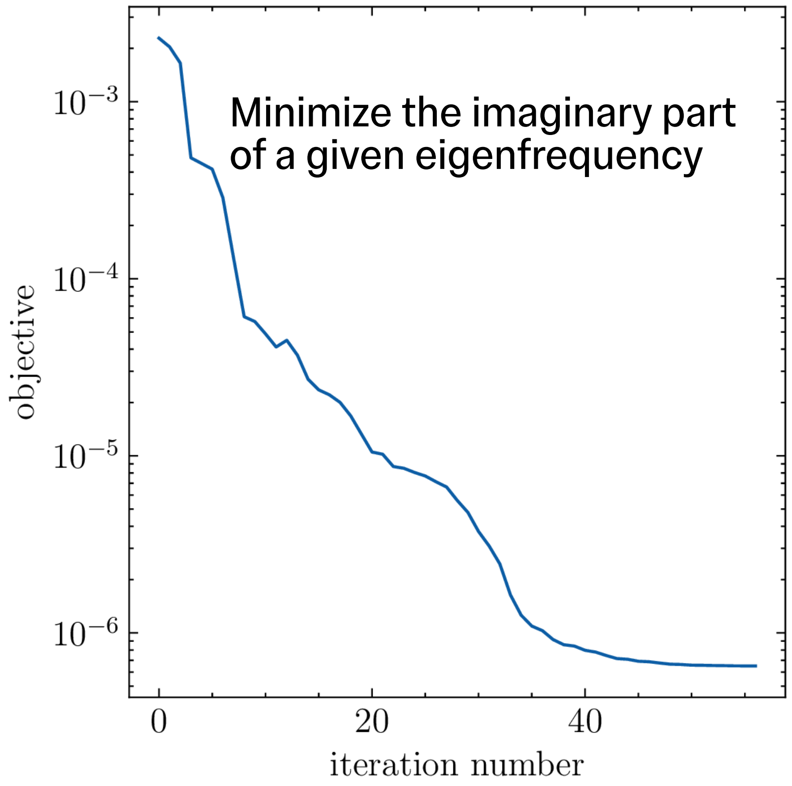
Hellmann–Feynman theorem
in quantum mechanics

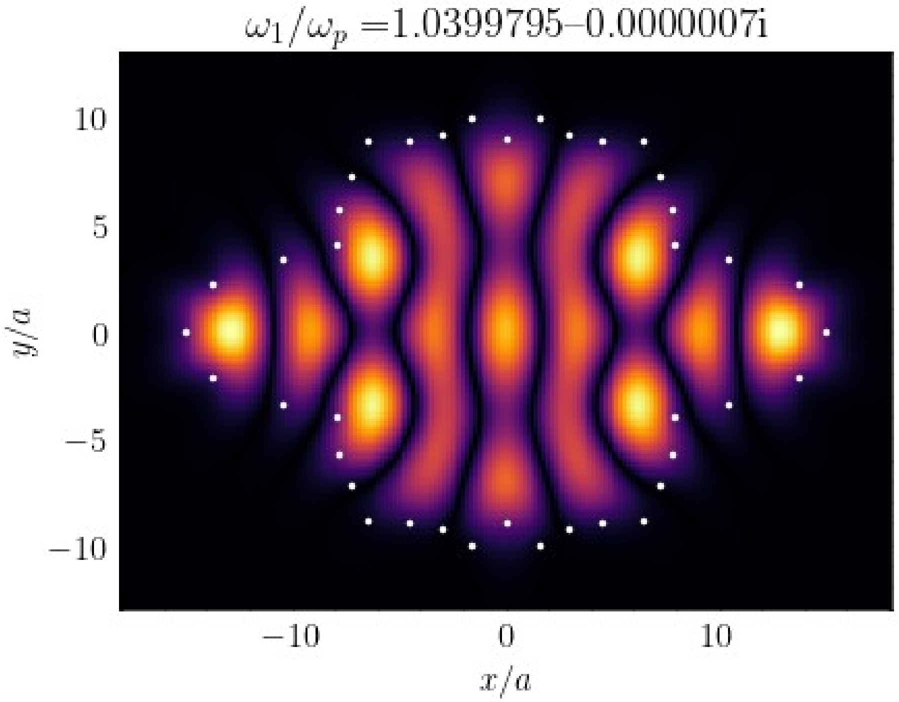
$$\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} = \left\langle \psi_\lambda \middle| \frac{\mathrm{d}\hat{H}_\lambda}{\mathrm{d}\lambda} \middle| \psi_\lambda \right\rangle$$



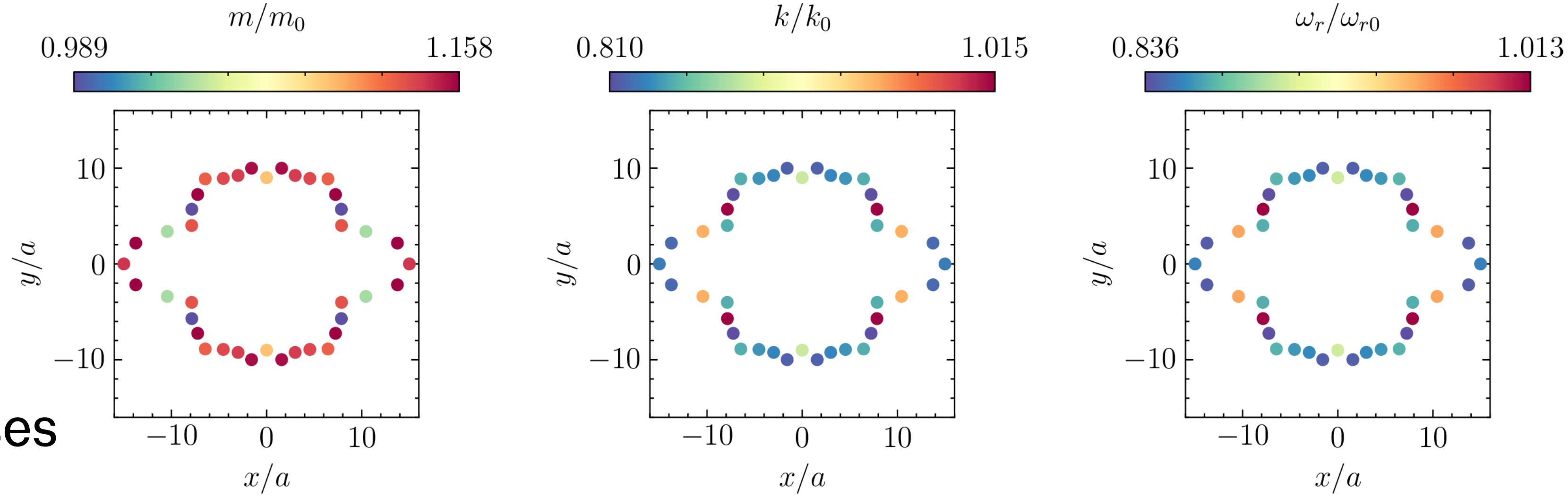
→ gradient based optimisation

Resonances engineering
Designing quasi BICs





Quality factor
$Q \simeq 7.3 \times 10^5$

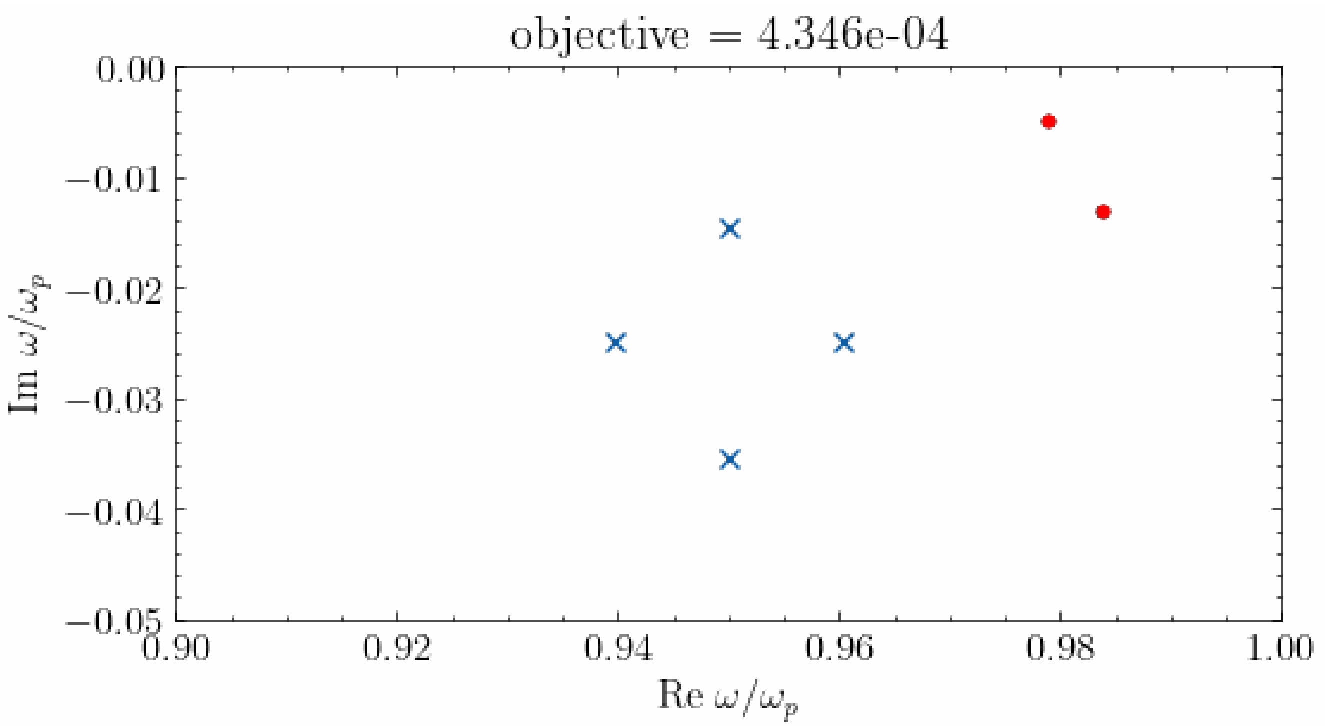


Optimizing masses and stiffnesses

Fixed positions

Resonances engineering

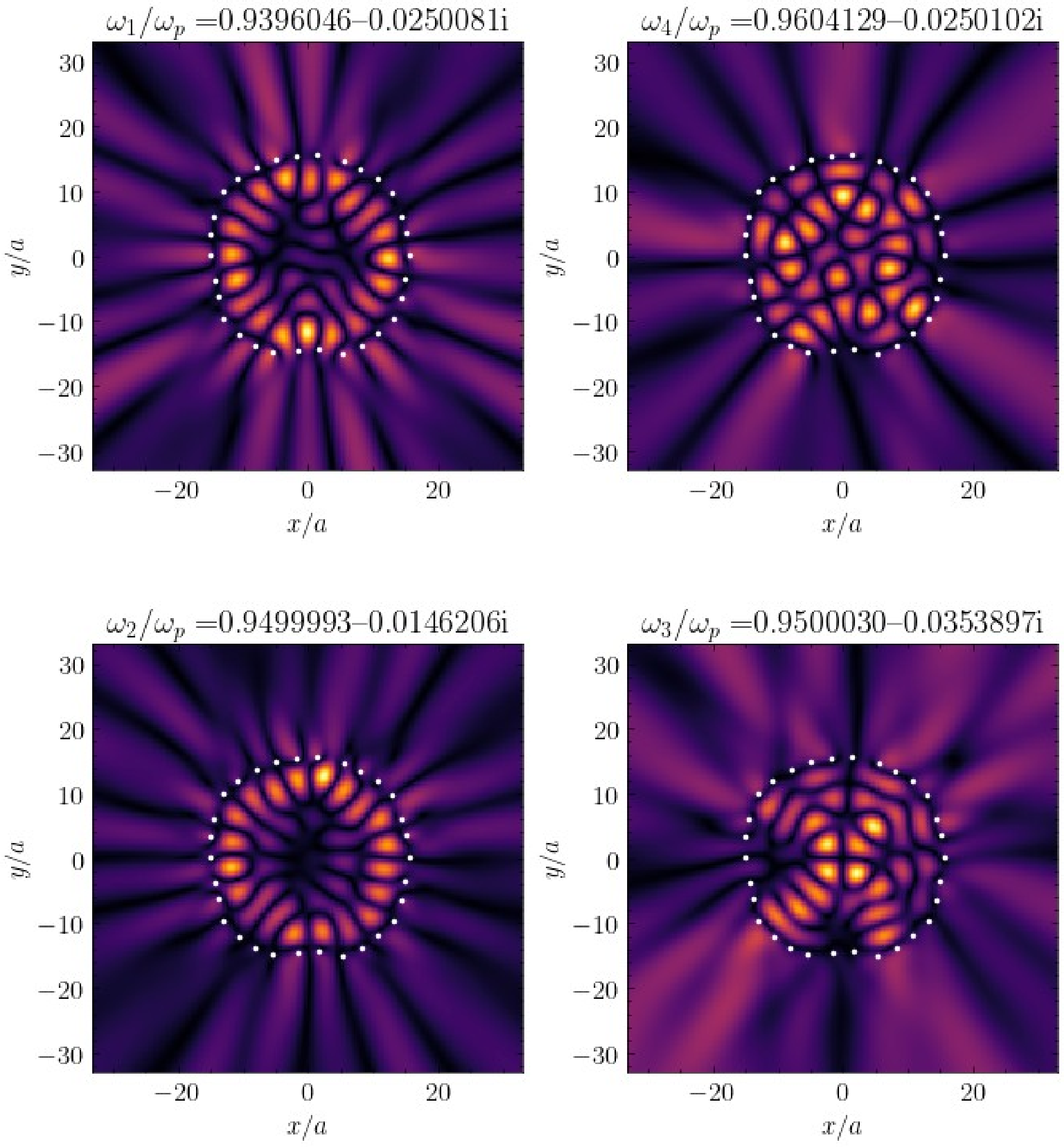
Placing eigenvalues in the complex plane



$$\mathcal{G}(\mathbf{R}_\alpha) = \sum_{n=1}^{4} \left|\omega_n(\mathbf{R}_\alpha) - \omega_n^{\mathrm{tar}}\right|^2 .$$

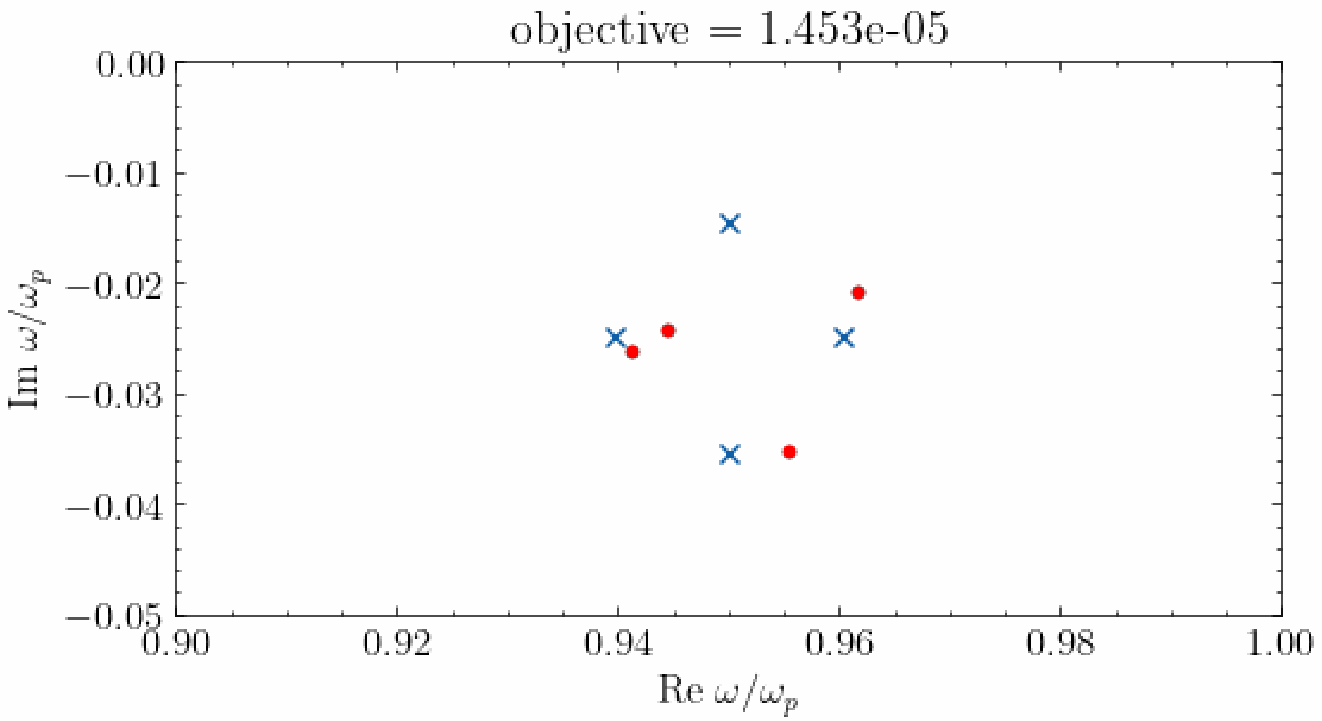
Optimizing resonators positions

Fixed masses and stiffnesses



Resonances engineering

Placing eigenvalues in the complex plane



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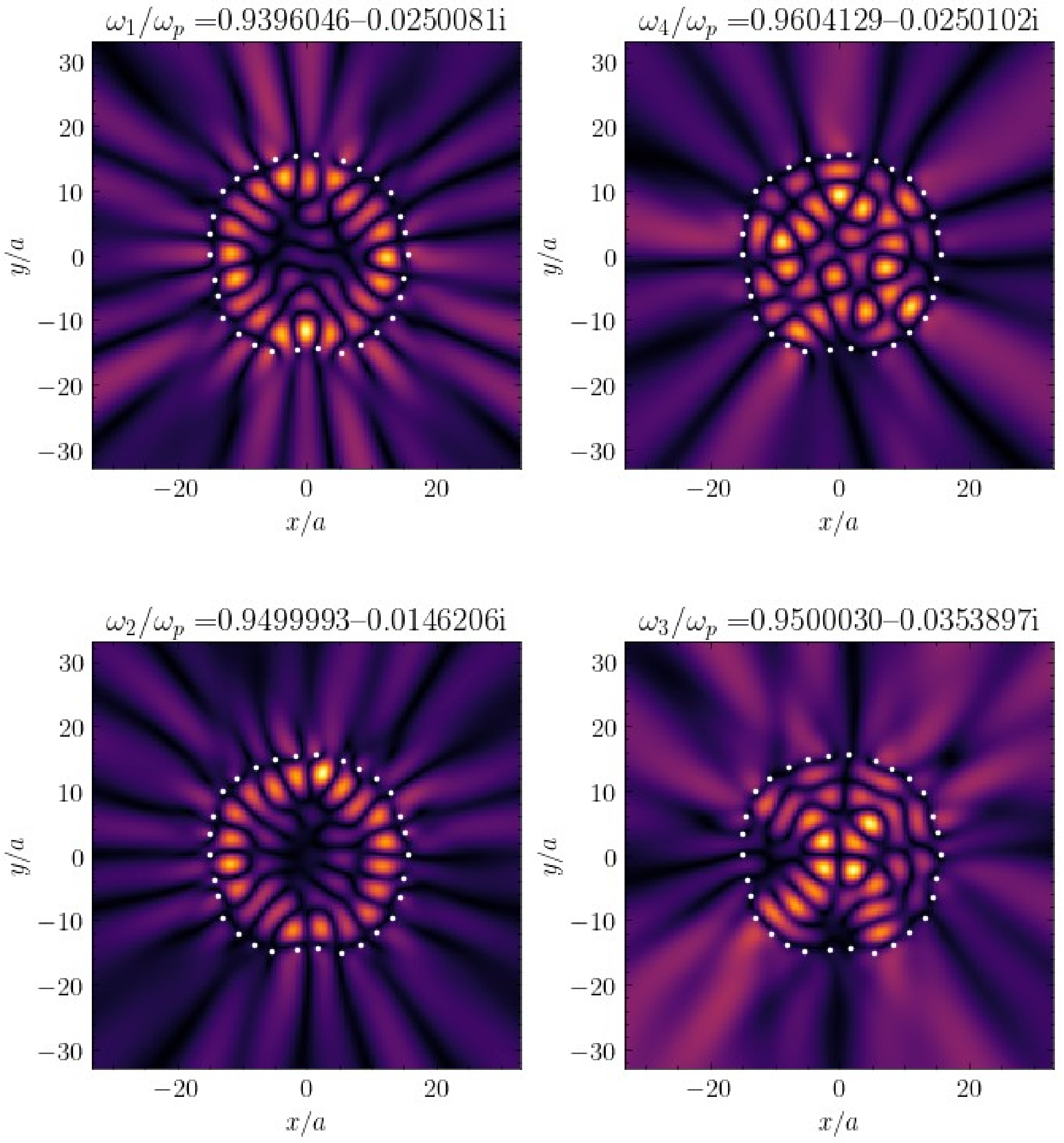
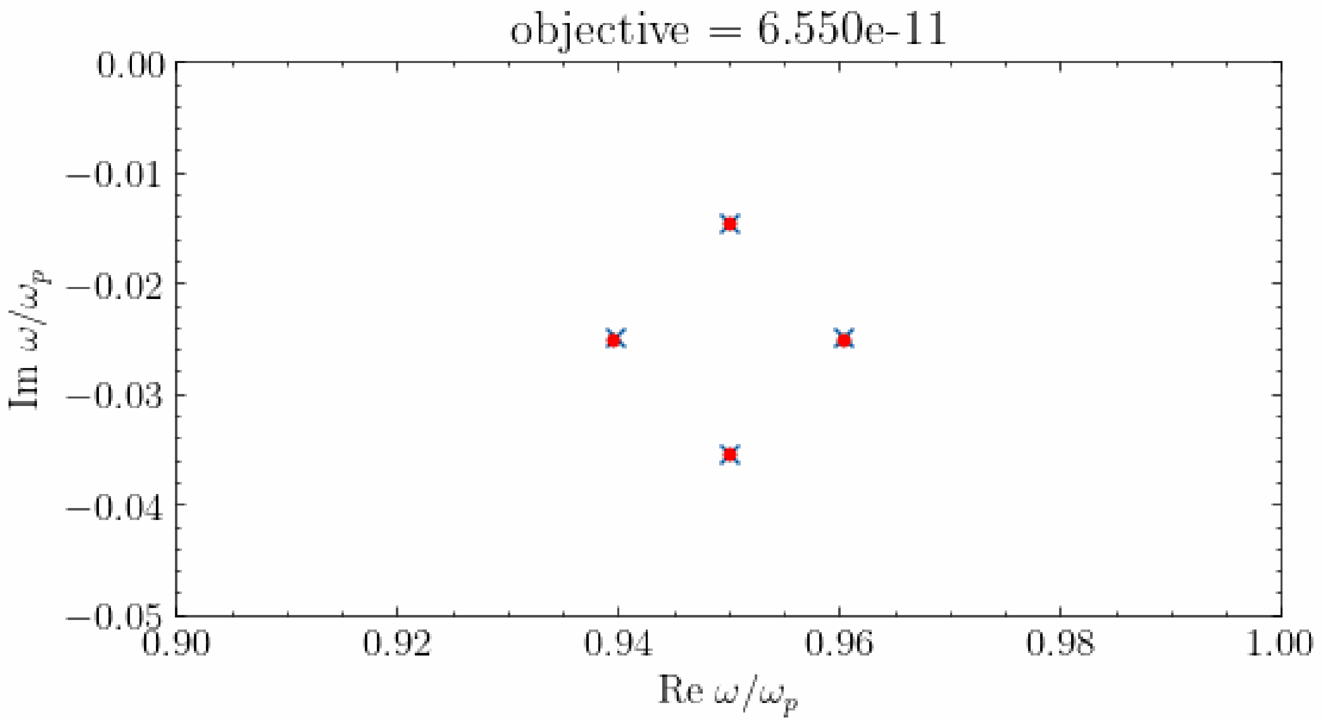
Optimizing resonators positions

Fixed masses and stiffnesses



Resonances engineering

Placing eigenvalues in the complex plane



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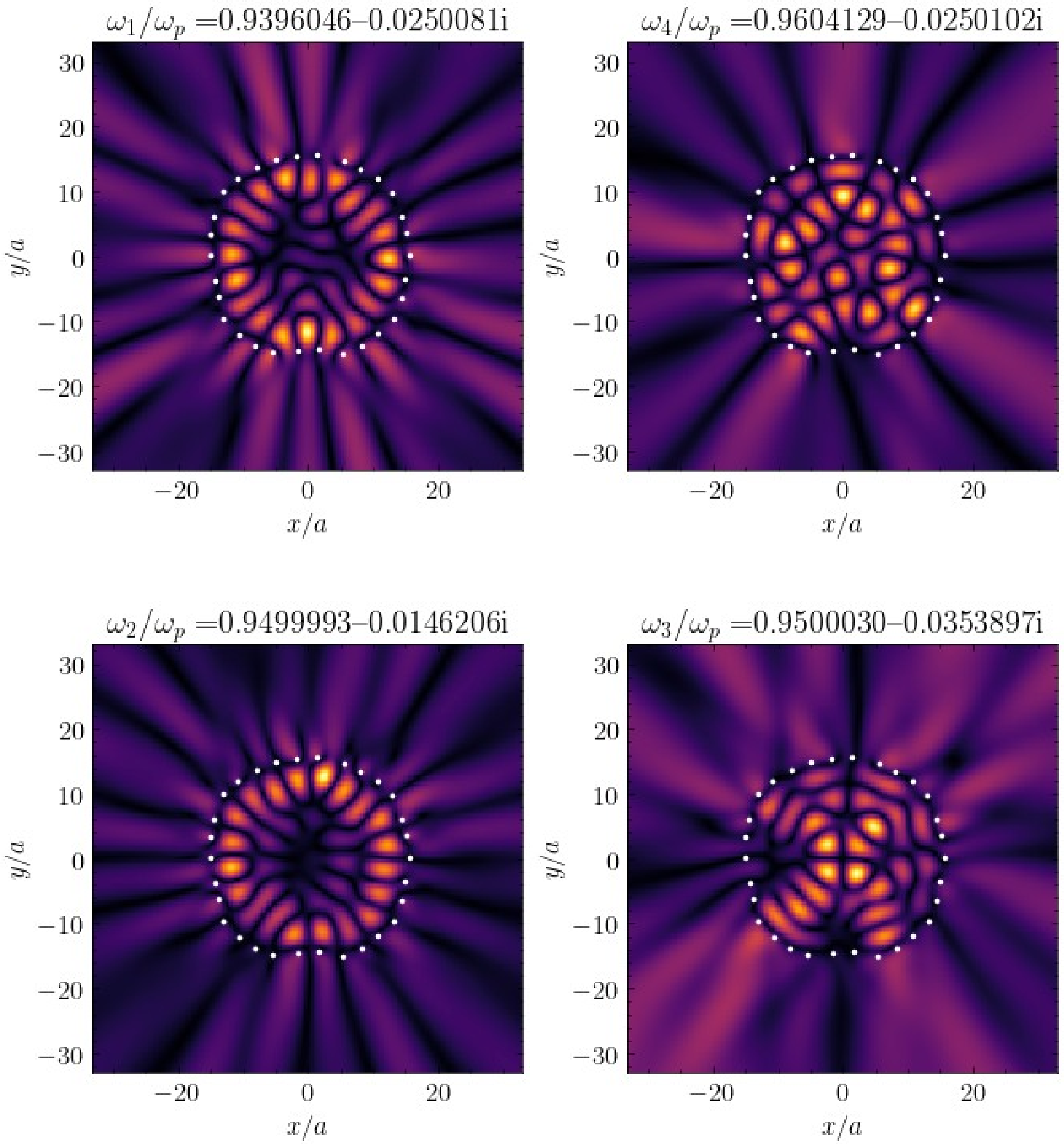
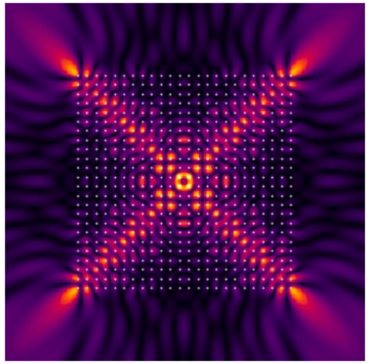


Optimizing resonators positions

Fixed masses and stiffnesses

Conclusion

Conclusion and future work



Open source Python package:
https://benvial.gitlab.io/klove/

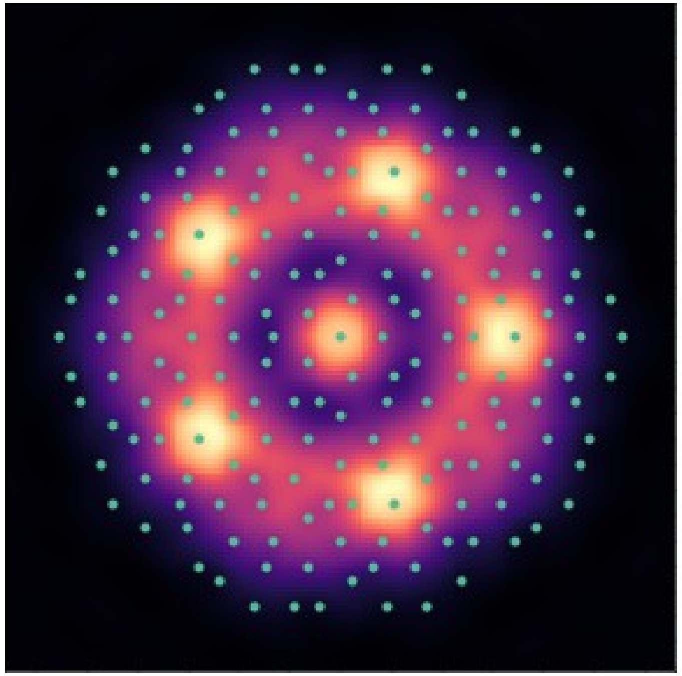
QNM analysis and expansion

Fast reduced order model with a few modes

Physical insight into the resonant interraction of sources with modes

Resonance engineering and optimization

Extension: gratings



Vial, B., Sabaté, M. M., Wiltshaw, R., Guenneau, S. & Craster, R. V. Platonic quasi-normal modes expansion. (2024)

Preprint at https://doi.org/10.48550/arXiv.2407.12042

IMPERIAL

Thank you

Modal Analysis for Controlling Elastic Waves in Platonic Metamaterials

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09/09/2024

METAMATERIALS ENHANCED VIBRATION ENERGY HARVESTING

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Contour integrals

$$f(\omega) = \sum_{n\in\mathbb{N}} \frac{A_n}{(\omega-\omega_n)^p} + g(\omega)$$

Γ closed loop containing the pole ω_m only. Defining the integrals I_k for $k=0,1,2$:

$$I_k = \frac{1}{2i\pi}\oint_\Gamma \omega^{k+p-1} f(\omega)\mathrm{d}\omega = \frac{1}{2i\pi}\oint_\Gamma \omega^{k+p-1}\frac{A_m}{(\omega-\omega_m)^p}\,\mathrm{d}\omega$$

since by Cauchy's theorem, the integral of g along a closed path is null.
Applying the residue theorem to $f_k : \omega \mapsto \omega^{k+p-1}\frac{A_m}{(\omega-\omega_m)^p}$ one obtains:

$$I_k = \mathrm{Res}_{\omega_m} f_k = \frac{1}{(p-1)!}\lim_{\omega\to\omega_m}\frac{\partial^{p-1}}{\partial\omega^{p-1}}\left[(\omega-\omega_m)^p f_k(\omega)\right] = A_m\frac{(k+p-1)!}{k!(p-1)!}\omega_m^k$$

We define $r_{01} = \frac{(p-1)!}{p!}\frac{I_1}{I_0}$ and $r_{12} = 2\,\frac{p!}{(p+1)!}\frac{I_2}{I_1}$.
Hence $A_m = I_0$, and $\omega_m = r_{01} = r_{12}$.
Three cases:

- no poles if $I_0 = I_1 = 0$
- a single pole if $r_{01} = r_{12}$
- several poles if $r_{01} \neq r_{12}$

Rayleigh quotient

Choose an initial pair $\left(\omega^{(0)}, \Psi^{(0)}\right)$ with $\left\|\Psi^{(0)}\right\| = 1$ and a nonzero vector Φ.
for $k = 0, 1, \dots$ until convergence do

Solve

$$M\left(\omega^{(k)}\right)\widetilde{\Psi}^{(k+1)} = M'\left(\omega^{(k)}\right)\Psi^{(k)} \text{ for } \widetilde{\Psi}^{(k+1)}.$$

Set

$$\omega^{(k+1)} = \omega^{(k)} - \frac{\Phi\Psi^{(k)}}{\Phi\widetilde{\Psi}^{(k+1)}}.$$

Normalize

$$\Psi^{(k+1)} = \frac{\widetilde{\Psi}^{(k+1)}}{\left\|\widetilde{\Psi}^{(k+1)}\right\|}.$$

Explicit expressions for gradients

Explicit expressions considering the resonator γ are given by the following:

$$\frac{\partial \omega_n}{\partial m_{R\gamma}} = -\frac{D}{m_{R\gamma}^2 \omega_n^2} \Phi_{n,\gamma}^2,$$

$$\frac{\partial \omega_n}{\partial k_{R\gamma}} = -\frac{D}{k_{R\gamma}^2} \Phi_{n,\gamma}^2.$$

The derivative of a matrix element $M_{\alpha\beta}$ with respect to position x_γ (a similar expression holds for the y_γ coordinates) is zero unless ($\alpha = \gamma$ or $\beta = \gamma$) and $\alpha \neq \beta$, we then have:

$$\frac{\partial M_{\alpha\beta}}{\partial x_\gamma}(\omega_n) = \xi \frac{x_\alpha - x_\beta}{|\mathbf{R}_\alpha - \mathbf{R}_\beta|} k_n G_1(\mathbf{R}_\alpha - \mathbf{R}_\beta),$$

with $\xi = 1$ if $\alpha = \gamma$ and $\xi = -1$ otherwise, and $G_1(\mathbf{r}) = \frac{i}{8k^2}\left[H_1(kr) - iH_1(ikr)\right]$, where H_1 is the first-order Hankel function of the first kind.